Assignment : Differential equation

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QNo 1:- $x^3y''' + 2x^2y' + 2y = 10x + 10/x$

Sol:- $\frac{x^3 d^3 y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$

$x^3D^3y + 2x^2D^2 + 2y = 10x + 10x^{-1}$

let $x = et$

$t = \ln x$

$XD = D$

$X^2D^2 = D(D-1) - D^2 - D$

$X^3D^3 = D(D-1)(D-2)$

Now from this we get

$(D^3 - 3D^2 + 2D + 2(D^2 - D) + 2)y = 10^x + 10/x$

$(D^3 - 3D^2 + 2D + 2D^2 - 2D + 2)y = 10^x + 10/x$

$(D^3 - D^2 + 2)y = 10^x + 10^{-1}$

$D = m$

$x = e^t$

Putting in equation

$(m^3 - m^2 + 2)\, y = 10e^t + 10/e^t$

By synthetic division

$$
\begin{array}{r|rrrr}
 & 1 & -1 & 0 & 2 \\
-1 & & -1 & 2 & -2 \\
\hline
 & 1 & -2 & 2 & \boxed{0}
\end{array}
$$

So we have

$D^2 - 2D + 2 = 0$

using Quadratic formula

$a = 1 \qquad b = -2 \qquad c = 2$

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-(-2) \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)}$$

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-1}\,\sqrt{4}}{2}$$

$$D = \frac{2 \pm \sqrt{-1}\,(2)}{2}$$

$\sqrt{-1} = i$

$$D = \frac{2 \pm i(2)}{2}$$

$$D = \frac{2 \pm 2i}{2}$$

$$D = \frac{\cancel{2}(1+i)}{\cancel{2}}$$

$$D = 1+i$$

Now particular Method

$$y_p = \frac{1}{D^3 - D^2 + 2} \cdot 10e^t + \frac{1}{D^3 - D^2 + 2} \cdot \frac{10}{e^t}$$

$$y_p = \frac{10e^t}{D^3 - D^2 + 2} + \frac{10/e^t}{D^3 - D^2 + 2}$$

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$$y_p = \frac{10e^{t}}{(1)^2-(1)^2+2} + \frac{10^{-et}}{(1)^2-(1)^2+2}$$

$$y_p = \frac{10e^{t}}{2} + \frac{10^{-et}}{2}$$

$$y_p = 5e^{t} + 5^{-et}$$

General solution

$$y = e^{-x}(c_1 \cos t + c_2 \sin t) + 5e^{t} + 5^{-et}$$

put $e^{t} = x$ and $t = \ln x$

$$y = e^{-x}(c_1 \ln x + c_2 \sin x) + 5e^{x} + 5^{-e^{x}}$$

$$y = e^{-x}(c_1 \ln x + c_2 \sin x) + 5e^{x} + 5/e^{x}$$

Q2:- $x^3 y''' + 4x^2 y'' - 5xy' - 15y = x^4$

Sol:- $x^3 y''' + 4x^2 y'' - 5xy - 15y$

$y = x^m \qquad y' = mx^m$

$y'' = m(m-1)x^{m-2}$

$y''' = m(m-1)(m-2)x^{m-3}$

$= x^3 m(m-1)(m-2)x^{m-3} + 4m(m-1)x^m - 5mx^m - 15x^m = 0$

$m^3 \left[[m^2 - m](m-2) + 4(m^2 - m) - 5m - 15 \right] = 0$

$m^3 - 2m^2 - m^2 + 2m + 4m^2 - 4m^2 - 5m - 15 = 0$

$m^3 - m^2 - 7m - 15 = 0$

Putting Valued

$(-1)^3 + (-1)^2 - 7(-1) - 15 = 0$

$-1 + 1 + 7 - 15 = 0$

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$m = 2$

$8 + 4 - 14 - 15 = (-17)$

$m = -2$

$-8 + 4 + 14 - 15$

$-4 - 1 = -5$

$m = 3$

$m = 2 + i$

$m = 2 - i$

Q3:- $y'' + 2/x\, y' - 6/x^2\, y = 10$

Sol:-

$f(x) = 10$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} c_1 x^{-3} & c_2 x \\ -3c_1 x^{-4} & 2c_2 x \end{vmatrix}$$

$$= 2c_1c_2 x^{-2} + 3c_1c_2 x^{-2} = 5c_1c_2 x^{-2}$$

$$W_1 = \begin{vmatrix} 0 & y_1 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & c_2 x^2 \\ 10 & 2c_2 x \end{vmatrix} = -10 c_2 x^{+2}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} c_1 x^{-3} & 0 \\ -3c_1 x^{-4} & 10 \end{vmatrix} = 10c_1 x^3$$

$$u_1' = \frac{w_1}{w} = \frac{-10c_2x^2}{5c_1c_2x^{-2}} = \frac{-2}{c_1}x^4$$

$$u_2' = \frac{w_2}{w} = \frac{10c_1x^{-3}}{5c_1c_2x^{-2}} = \frac{2}{c_2}x^{-1}$$

So

$$v_1' = \frac{-2}{c_1}x^4$$

$$u_2' = \frac{2}{c_2}x^{-1}$$

For v_1' we have

$$v_1 = \frac{-2}{c_1}\int x^4\,dx$$

$$v_1 = \frac{-2}{c_1}\int x^2\,dx$$

$$v_1 = \frac{-2}{5c_1}x^5$$

$$v_2 = \frac{2}{c_2}\int x^1\,dx$$

$$v_2 = \frac{2}{c_2}\ln x$$

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Now particular integral

$$y_p = \frac{-2}{5c_1} x^5 c_1 x^3 + \frac{2}{c_2} \ln x \, c_2 x^2$$

$$y_p = -2x^2 + 2\ln(x) x^2$$

$$y = c_1 x^{-3} + c_2 x^2 - 2x^2 + 2\ln x (x^2)$$

Q4:-

$$x^2 y'' + 7xy' + 5y = x^5$$

$$y(0) = 2 \quad \text{and} \quad y'(1) = 2$$

Sol:-

$$x^2 \frac{dy^2}{dx^2} + 7x \frac{dy}{dx} + 5y = \cancel{x}^5$$

$$\left(x^2 \frac{d^2}{x^2} + 7 \frac{dxy}{dx} \right) = x^5$$

Put $xD = D$

$$x^2 D^2 = D^2 - D$$

$$= (D^2 + D + 7D + 5) y = e^{5t}$$

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$= (D^2 - D + 7D + 5)y = e^{5t}$

$(D^2 + 6D + 5)\ y = e^{5t}$

By quardatic formula

$$D = \frac{-b \pm \sqrt{-b^2 - 4ac}}{2a}$$

$$D = \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$D = \frac{-6 \pm \sqrt{16}}{2}$$

$$D = -3 \pm 2$$

The following are the roots

$y_c = C_1 e^{-5t} + C_2 e^{-t}$

As $y_p = \frac{1}{D^2 + 6D + 5} e^{5t}$

$$y_p = \frac{1}{(5)^2 + 6(5) + 2} e^{5t}$$

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$y_p = \frac{1}{60} \cdot e^{5t}$

General solution

$y = y_2 + y_p$

$y = c_1 x^{-5} + c_2 x^{-1} + \frac{1}{60} e^{5t}$

Now for $x = 0$

$e^{0} = 1$

$y(0) = 2 \qquad y = 2$

$2 = c_1 (2)^{-5} + c_2 (2)^{-1} + \frac{1}{60} (2)^5$

$2 = -32 c_1 - 2 c_2 + \frac{8}{15}$

$2 - 8/15 = -32 c_1 - 2 c_2$

$\frac{22}{15} = 3c_1 - 2 c_2$

Now differenciate the other

eqdation $\quad y(1) = 2x = 2$

$2 = -5 c_1 c_2^{-6} - c_2 (2)^{-2} + \frac{1}{12} (2)^4$

$2 = -5 c_1 (-64) - c_2 (4) + 16/12$

$2 = 320c_1 + 4c_2 + 4/3$

$2 - 4/3 = 320c_1 + 4c_2$

Multiply eq (2) and then integration it.

$-44/18 = 64$

$2/3 = \pm 320c_1 + 4c_2$

$34/15 = -256c_1$

$c_1 = 580$

Putting valued of eq (1)

$\frac{22}{15} + 8560 = -2c_2$

$c_2 = \frac{18561}{-2}$

$c_2 = -9280$

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Putting values of c_1 & c_2

$$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$$

Q5:- $(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$

$y = x^m$ $\quad y' = m x^{m-1}$

$y'' = m(m-1)x^{m-2}$

Sol:- $y = m$

$y' = m x^{m-1}$ $\quad y'' = m(m-1)x^{m-2}$

$= (x^2+1+2x)\, m(m-1)x^{m-2} - 3(x+1)\, m x^{m-1} + 4x^m = 0$

$x^2 m(m-1)x^{m-2} + m(m-1)x^{m-2} + 2x m(m-1)x^{m-2}$

$-3(x m x^{m-1} + m x^{m-1}) + 4x^m = 0$

$m(m-1)x^m - 3mx^m + 4x^m + 2m(m-1)x^{m-1} + m(m-1)x^{m-2}$

$(m(m-1) - 3m + 4)x^m + (2m(m-1)x^{m-1}) + m(m-1)x^{m-2}$

$m(m-1) - 3m + 4 = 0$

$m^2 - m - 3m + 4 = 0$

$m^2 - 4m + 4 = 0$

$a = 1 \quad b = -4 \quad c = 4$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$$

$$m = \frac{4 \pm \sqrt{0}}{2} = \frac{4}{2} = 2$$

$m = 2$

$y = c_1 x^2 + c_2 x^2 \ln x$

$y_1 = c_1 x^2 \quad y_2 = c_2 x^2 \ln x$

$$y'' - \frac{3y'}{(x+1)} + \frac{4}{(x+1)^2} = \frac{x^2}{(x+1)^2} = \left(\frac{x}{x+1}\right)^2$$

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$$f(x) = \frac{x^2}{(x+1)^2}$$

$$w = \begin{vmatrix} 4x^2 & c_2 x^2 \ln x \\ 2c_1 x & c_2 x + c_2 \ln x \end{vmatrix}$$

$$w_1 = \begin{vmatrix} 0 & c_2 x^2 \ln x \\ \frac{x^2}{(x+1)^2} & c_2 x + c_2\, 2x \ln x \end{vmatrix}$$

$$w_1 = \frac{-x^2}{(x+1)^2} \quad c_2 x^2 \ln x$$

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$w_2 = \begin{vmatrix} c_1 x^2 & 0 \\ 2c_1 x & \frac{x^2}{((x+1)^2} \end{vmatrix}$$

$$w_2 = \frac{c_1 x^4}{(cx+1)^2}$$

$$u = \frac{w_1}{w}$$

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$$u_2' = \frac{w_2}{w}$$

$$y_p = u_1 p_1 + u_2 p_1$$

$$y = y_n + y_p$$