Name: Hassan Mehdi
ID: 15453
Teacher: Sir. Latif Jan
Program: BC (CS)
Subject: Differential Equations
Midterm-Assignment
Course Code (CS): CSC-201
EDP Code (CS): 102002077
Spring Semester 2020

Differential Equations

Midterm-Assignment

Date: ....

Q1: Define Differential Equations with two examples.

Diffental Equation:

A Differential Equation is an equation that involves functions and its derivitives. The Solution to a differential equation is also a function or class of functions.

i.e

A Differential Equation of degree 2 Can be

$$y'' + 2y' = 3y \quad \text{or} \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3y$$

A Differential Equation of degree 4 Can be

$$\left(\frac{dy}{dx}\right)^3 + \frac{d^4y}{dy^4} + y = 2\sin(x)\cos^3(x)$$

(b): Define Seperable Differential Equation:

Seperable Differential Equation:

A Differential Equation that can be factored into two parts is called a Seperable Differential Equation.

i.e

$$G(x,y) = M(x)\,N(y) \text{ form.}$$

If we had an equation

$$\frac{dy}{dx} = \frac{-x}{ye^{x^2}}$$

This equation is a seperable Differential Equation because we can factor it in this way

$$y\,dy = -x\bar{e}^{x}\,dx$$

Date: ....

i: Solve the following Seperable Differential equations.

(a) $y' = \frac{xy^3}{\sqrt{1+x^2}}$, $y(0) = -1$

<u>Sol</u>

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

Multiply $\frac{1}{y^3}$ B/S

$$\frac{1}{y^3} \times \frac{dy}{dx} = \frac{x\cancel{y^3}}{\sqrt{1+x^2}} \times \frac{1}{\cancel{y^3}}$$

Multiply dx B/S

$$\frac{1}{y^3} \frac{dy}{\cancel{dx}} \times \cancel{dx} = x(1+x^2)^{-\frac{1}{2}} dx$$

Integrate on B/S

$$\int y^{-3} dy = \int x(1+x^2)^{-1/2} dx$$

$$\frac{1}{2y^2} = (1+x^2)^{1/2} + C \quad — (1)$$

Put $y(0) = -1$ in (1)

Darsi Notes

$$\cancel{\frac{1}{2(1)^2}} \quad \frac{-1}{2(1)^2} = (1+0^2)^{1/2} + c$$

$$\frac{-1}{2} = 1 + c$$

$$\frac{-1}{2} - 1 = c = \frac{-3}{2} \quad \text{— (ii)}$$

Put C in (i)

$$\frac{-1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

for $y(x)$

$$\frac{-1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

Take $\sqrt{\ }$ B/S

$$y = \pm \frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Sign is -ve from initial condition.

$$y = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Date:

Now finding interval of validity

$$y(x) = \frac{1}{-\sqrt{3-2\sqrt{1+x^2}}}$$

$$3-2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4\sqrt{1+x^2}$$

$$\frac{9}{4} > 1+x^2$$

$$\frac{5}{4} > x^2$$

$$\frac{-\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

(b)

$$y' = e^{-y}(2x-4), \quad y(5)=0$$

$$\frac{dy}{dx} = e^{-y}(2x-4)$$

Mutiply e^{y} B/S

$$e^{y}\frac{dy}{dx} = \cancel{e^{y}} \cdot \cancel{e^{-y}}(2x-4)$$

Multiply dx B/S

$$e^{y}\frac{dy}{\cancel{dx}} \times \cancel{dx} = (2x-4)\,dx$$

$$e^{y}dy = (2x-4)\,dx$$

Integrate on B/S

$$\int e^{y}dy = \int (2x-4)\,dx$$

$$e^{y} = x^2 - 4x + C \quad -\text{(i)}$$

At $y(5)=0$ in (i)

$$e^{0} = 5^2 - 4(5) + C$$

$$1 = 25 - 20 + C$$

$$1 = 5 + C \Rightarrow C = -4 \quad -\text{(ii)}$$

Date:

Put C in (i)

$\therefore e^{y} = x^2 - 4x - 4$

for $y(x)$

$$y(x) = \ln(x^2 - 4x - 4)$$

$$x^2 - 4x - 4 > 0$$

Possible Intervals of Validity are

$$\infty < x < 2 - 2\sqrt{2}$$

$$2 + 2\sqrt{2} < x < \infty$$

Q2: Solve the following Differential Equations using Linear method.

(i)

Steps for Solving Linear D.E:-

Following are the steps of solving a linear Differential Equation.

- Convert the given Equation into

$$\frac{dy}{dx} + P(x)y = q(x) \text{ form.}$$

- Find the integration factor

$$V(x) = e^{\int P(x)dx}$$

- Evaluate the integrals

$$\int v(x)q(x)dx$$

- Find the General equation

$$\frac{\int v(x)q(x)dx + C}{v(x)}$$

- Complete the Constant "C" using initial Condition.

Date: ....

(ii) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1,$

$$y\left[\frac{\pi}{4}\right] = 3\sqrt{2}, \quad 0 \le x \le \left[\frac{\pi}{2}\right].$$

Sol

$$y' + \frac{1}{\cos(x)} \cdot \sin(x)y = 2\cos^2(x) - \cos(x)\sin(x) - \frac{1}{\cos(x)}$$

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \sec(x)$$

Finding Integral Factor.

$$\int \tan(x)dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1}$$

$$= \ln|\sec(x)|$$

Multiply Integral factor through the equation.

$$\sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$= (\sec(x)y)' = 2\cos(x)\sin(x) - \sec^2(x).$$

Integrate on B/S

$$\int (\sec(x)y(x))'dx = \int 2\cos(x)\sin(x) - \sec^2(x)dx$$

$\Rightarrow \sec(x)\,y(x) = \frac{-1}{2}\cos(2x) - \tan(x) + C$

$y(x) = \frac{-1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + C\cos(x)$

$y(x) = \frac{-1}{2}\cos(x)\cos(2x) - \sin(x) + C\cos(x)$ — (i).

Put $y\left[\frac{\pi}{4}\right] = 3\sqrt{2}$ in (i)

$3\sqrt{2} = y\left[\frac{\pi}{4}\right] = \frac{-1}{2}\cos\left[\frac{\pi}{4}\right]\cos\left[\frac{\pi}{2}\right] - \sin\left[\frac{\pi}{4}\right] + \cos\left[\frac{\pi}{4}\right]$

$3\sqrt{2} = \frac{-\sqrt{2}}{2} + C\frac{\sqrt{2}}{2}$

$C = 3\sqrt{2} + \frac{\sqrt{2}}{2} \times \frac{2}{\sqrt{2}}$

$C = 7$ — (ii)

Put C in (i)

$y(x) = \frac{-1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$

Date:

(iii) $u' + 2u = \sin t$

Sol $\frac{du}{dt} + 2u = \sin t.$

Finding the integration factor.

$$e^{\int 2\,du} = 2u$$

Multiplying the Integration factor through the equation

$$u = \frac{1}{e^{2u}}\left[\int e^{2u}\cdot(\sin t) + C\right]$$

$$u = \frac{1}{e^{2u}}\left[\frac{e^{3u}}{3}\cdot(-\cos t) + C\right]$$

Multiply $\frac{1}{e^{2u}}$ inside.

$$u = \left[\frac{1}{e^{2u}}\cdot\frac{e^{3u}}{3} - \frac{1}{e^{2u}}\cos t + \frac{1}{e^{2u}}\cdot C\right]$$

$$u = \frac{1}{3}e^{u} - \frac{\cos t}{e^{2u}} + \frac{C}{e^{2u}}$$

$$u = \frac{1}{3}e^{u} - \cos t\, e^{-2u} + C e^{-2u}$$

Date: ....

Q3: Solve the following IVP for Exact equation and find the interval of validity for the Solution.

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \ y(0) = -3$

Sol

Identify M and N.

$M = 2xy - 9x^2, \quad M_y = 2x$

$N = 2y + x^2 + 1, \quad N_x = 2x$

$M_y = N_x$, Hence the EQ is Exact.

$\Psi_x = M \quad , \quad \Psi_y = N$

$\frac{d}{dx}\Psi = 0$

$\Psi_x = 2xy - 9x^2$

Integrate on B/S

$\int dx\, \Psi_x = \int 2xy - 9x^2\, dx + f(y)$

$$\psi = x^2 y - 3x^3 + f(y)$$

$$\frac{\partial \psi}{\partial y} = x^2 + f'(y) = 2y + x^2 + 1 = N$$

$$f'(y) = 2y + 1$$

$$f(y) = \int 2y + 1\,dy = y^2 + y + c$$

$$\psi(x, y) = x^2 y - 3x^3 + y^2 + y + C$$

$$\psi(x, y) = y^2 + (x^2+1)y - 3x^3 + C$$

$$\frac{d\psi(x,y)}{dx} = 0, \quad \psi(x,y) = C$$

$$y^2 + (x^2+1)y - 3x^2 = C \quad -(1)$$

Put $y(0) = -3$ in (1)

$$(-3)^2 + (0+1)(-3) - 3(0)^3 = C$$

$$6 = C \quad \text{or} \quad C = 6$$

$$\Rightarrow \quad y^2 + (x^2+1)y - 3x^3 - 6 = 0$$

for $y(x)$

Date: ....

$$y(x) = \frac{(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^2-6)}}{2(1)}$$

$$y(x) = \frac{(x^2+1) \pm \sqrt{x^4+12x^3+2x^2+25}}{2}$$

$$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = -3, 2$$

From initial Condition Sign is -ve

$$y(x) = \frac{-(x^2+1) - \sqrt{x^4+12x^3+2x^2+25}}{2}$$

Finding interval of validity

$$x^4+12x^3+2x^2+25=0$$

Possible intervals are

$$-\infty < x \leq -11.8$$
$$-1.39 \leq x < \infty$$

Darsi Notes

(ii) $\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0, \quad y(5) = 0$

Sol

Re-write Equation

$$\frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)\,y' = 0$$

Finding M & N.

$$M = \frac{2ty}{t^2+1} - 2t, \qquad M_y = \frac{2t}{t^2+1}$$

$$N = \ln(t^2+1) - 2, \qquad N_t = \frac{2t}{t^2+1}$$

$M_y = N_t$, hence Equation is exact.

$$\psi(t,y) = \frac{2ty}{t^2+1} - 2t$$

Integrate on B/S

$$\psi(t,y) = \int \frac{2ty}{t^2+1} - 2t\,dt$$

$$\psi(t,y) = y\ln(t^2+1) - t^2 + f(y)$$

$$\psi_y = \ln(t^2+1) + f'(y)$$

Date: ....

$\psi_y = \ln(t^2+1) - 2 = N$

$f'(y) = -2, \quad f(y) = -2y$

$\psi(t,y) = y\ln(t^2+1) - t^2 - 2y$

$y\ln(t^2+1) - t^2 - 2y = c \quad -(i)$

Put $y(5)=0$ in (i)

$0\ln(5^2+1) - 5^2 - 2(0) = 0$

$-25 = C$ or $C = -25$

$\Rightarrow \quad y(\ln(t^2+1)-2) - t^2 = -25$

for $y(t)$

$$y(t) = \frac{t^2-25}{\ln(t^2+1)-2}$$

$\ln(t^2+1) - 2 = 0$

$\ln(t^2+1) = 2$

$t^2+1 = e$

Darsi Notes

Possible intervals are

$$-\infty < t < -\sqrt{e^2-1}$$

$$-\sqrt{e^2-1} < t < \sqrt{e^2-1}$$

$$\sqrt{e^2-1} < t < \infty$$