FINAL TERM EXAMINATION

NAME SABJ-UL-HASSAN

ID 7932

SECTION B

DEPARTMENT BE(CIVIL)

SUBJECT DEFFERENTIAL EQUATION

SUBMITTED TO MAM. SHOMAILA MAZHAR

DATE 30 JUNE 2020

QUESTION No 1

The Wavelength Equation :~

We generally visit beach .... relevant partial derivative?

PART (i)

$$W = \sin(x+ct) + \cos(2x+2ct)$$

SOLUTION :~

As we know that

$$w = \sin(x+ct) + \cos(2x+2ct)$$

Now taking partial derivative w.r.t 't'

$$\frac{\delta w}{\delta t} = \frac{\delta}{\delta t}(\sin(x+ct)) + \frac{\delta}{\delta t}(\cos(2x+2ct))$$

$$\Rightarrow \frac{\delta w}{\delta t} = \cos(x+ct).c + (-\sin(x+ct)).$$

$$\Rightarrow \frac{\delta w}{\delta t} = c(\cos(x+ct) - 2c\sin(2x+2ct)$$

$$\Rightarrow \frac{\delta w}{\delta t} = c(\cos(x+ct) - 2\sin(2x+2ct))$$

Again taking derivative

$$\Rightarrow \frac{\delta^2 w}{\delta t^2} = c\left\{\frac{\delta}{\delta t}\cos(x+ct) - 2\frac{\delta}{\delta t}\sin(2x+2ct)\right\}$$

$$\Rightarrow \frac{\delta^2 w}{\delta t^2} = c\left(-\sin(x+ct).c - 2\cos(2x+2ct).2c\right)$$

$$\Rightarrow \frac{\delta^2 w}{\delta t} = c\left\{-c\sin(x+ct) - 4c\cos(2x+2ct)\right\}$$

$$\Rightarrow \frac{\delta^2 w}{\delta t^2} = c^2\left\{-\sin(x+ct) - 4\cos(2x+2ct)\right\}$$

Now

$$w = \sin(x+ct) + \cos(2x+2ct)$$

Taking derivative w.r.t x

$$\Rightarrow \frac{\delta w}{\delta x} = \frac{\delta}{\delta x}\sin(x+ct) + \frac{\delta}{\delta x}\cos(2x+2ct)$$

$$\Rightarrow \frac{\delta w}{\delta x} = \cos(x+ct) + (-\sin(2x+2ct).2)$$

$$\frac{\delta w}{\delta x} = \cos(x+ct) - 2\sin(2x+2ct)$$

Now again taking derivative

$$\Rightarrow \frac{\delta^2 w}{\delta x^2} = \frac{\delta}{\delta x}\left(\cos(x+ct) - 2\frac{\delta}{\delta x}\sin(2x+2ct)\right)$$

$$\Rightarrow \frac{\delta^2 w}{\delta x^2} = -\sin(x+ct)\cdot 1 - 2\cos(2x+2ct)\cdot 2$$

$$\Rightarrow \frac{\delta^2 w}{\delta x^2} = -\sin(x+ct) - 4\cos(2x-2ct)$$

Now, As we know the wave equation

$$\frac{\delta^2 w}{\delta t^2} = c^2 \frac{\delta w}{\delta x}$$

Putting values

$$\Rightarrow \cancel{c^2}\{-\sin(x+ct) - 4\cos(2x+2ct)\} = \cancel{c^2}\{-\sin(x+ct) - 4\cos(2x+2ct)\}$$

$$\Rightarrow 1 = 1$$

L.H.S = R.H.S

Hence it is the proof of wave equation

QUESTION No 1 :~
PART B :~

$$w = \tan(2x + ct)$$

SOLUTION :~

As we know that

$$w = \tan(2x + ct) \quad \text{(1)}$$

Taking derivative w.r.t "t"

$$\Rightarrow \frac{\delta w}{\delta t} = \frac{\delta}{\delta t} \tan(2x + ct) \cdot c$$

$$\Rightarrow \frac{\delta w}{\delta t} = \sec^2(2x + ct) \cdot c$$

$$\Rightarrow \frac{\delta w}{\delta t} = c \sec^2(2x + ct)$$

Again taking derivative

$$\Rightarrow \frac{\delta^2 w}{\delta t^2} = c \frac{\delta}{\delta t} \sec^2(2x + ct)$$

$$\Rightarrow \frac{\delta^2 w}{\delta t^2} = c \cdot 2 \sec(2x + ct) \cdot \sec(2x + ct) \cdot \tan(2x + ct) \cdot c$$

Scanned with CamScanner

$$\Rightarrow \frac{\delta^2 w}{\delta t^2} = 2c^2 \sec^2(2x+ct) - \tan(2x+ct)$$

Now $\quad w = \tan(2x+ct)$

Taking derivative w.r.t x

$$\Rightarrow \frac{\delta w}{\delta x} = \frac{\delta}{\delta x} \tan 2x + ct$$

$$\Rightarrow \frac{\delta w}{\delta x} = \sec^2(2x+ct) \cdot 2$$

$$= \frac{\delta w}{\delta x} = 2 \sec^2(2x+ct)$$

Again taking derivative

$$\Rightarrow \frac{\delta^2 w}{\delta x^2} = 2 \frac{\delta}{\delta x} \sec^2(2x+ct)$$

$$\Rightarrow \frac{\delta^2 w}{\delta x^2} = 2.2 \sec(2x+ct) \sec(2x+ct) \tan(2x+ct) \cdot 2$$

$$\Rightarrow \frac{\delta^2 w}{\delta x^2} = 6 \sec^2(2x+ct) \tan(2x+ct)$$

As we know the wave equation

$$\frac{\delta^2 w}{\delta t^2} = c^2 \frac{\delta^2 w}{\delta x^2}$$

Putting values in wave equation

$$\Rightarrow \overset{1}{\cancel{2}}\,\cancel{c^2}\,\cancel{\sec^2(2x+ct)\tan(2x+ct)} = \cancel{c^2}\,\overset{3}{\cancel{6}}\,\cancel{\sec^2(2x+ct)\tan(2x+ct)}$$

$$1 \neq 3$$

So we have concluded that

$$L.H.S \neq R.H.S$$

Hence it is not the proof of wave equation

QUESTION No 2

Expand the following function in a Fourier Series

$$F(x) = x \quad , \quad -\pi < x \le 0$$
$$= 2x \quad , \quad 0 \le x \le \pi$$

SOLUTION :~

As we know that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum^{\infty} b_n \sin nx \quad \text{—(i)}$$

Here we know that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Now

$$\Rightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} x \, dx + \frac{1}{2\pi} \int_{0}^{\pi} 2x \, dx$$

$$\Rightarrow a_0 = \frac{1}{2\pi} \left[\frac{x^2}{2} \right]_{\pi}^{0} + \frac{1}{2\pi} (\cancel{2}) \left| \frac{x^2}{\cancel{2}} \right|_{0}^{\pi}$$

$$\Rightarrow a_0 = \frac{1}{2\pi} \left[0 - \frac{\pi^2}{2} \right] + \frac{1}{2\pi} \left[x^2 - 0 \right.$$

$$\Rightarrow a_0 = \frac{\pi^{\cancel{2}}}{4\cancel{\pi}} + \frac{1}{2\cancel{\pi}} (\pi^{\cancel{2}})$$

$$\Rightarrow \boxed{a_0 = \frac{1}{4}\pi + \frac{1}{2}\pi}$$

Now

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\cos nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\cancel{\frac{x\sin nx}{n}}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\left(\frac{dx}{dx}\cdot\frac{\sin(nx}{n}\,dx\right)\right]$$

$$+\frac{2}{\pi}\left[\cancel{\frac{x\sin nx}{n}}\Big|_{0}^{\pi} - \int_{0}^{\pi}\left(\frac{dx}{dx}\cdot\frac{\sin nx}{n}\,dx\right)\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left(\frac{\cos nx}{n}\Big|_{-\pi}^{0}\right) + \frac{2}{\pi}\left(\frac{\cos nx}{n}\Big|_{0}^{\pi}\right)$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[(\cos n(0) - \cos n(-\pi)\right] + \frac{2}{n\pi}\left[\cos n\pi - \cos n(0)\right]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}(1+1) + \frac{2}{n^2\pi}(-1-1)$$

$$a_n = \frac{2}{n^2\pi} + \frac{(-4)}{n^2\pi}$$

$$\boxed{a_n = \frac{-2}{n^2\pi}}$$

Scanned with CamScanner

Now

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)\Big|_{-\pi}^{0} - \int_{-\pi}^{\pi}\left(\frac{d}{dx}x\left(\frac{-\cos nx}{n}\right)dx\right.\right.$$
$$+\frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\Big|_{0}^{\pi}\right) - \int_{0}\left[\frac{d}{dx}x\left(\frac{-\cos nx}{n}\right)dx\right]\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}\left(\frac{x\,\cancel{\pi}(2)}{\cancel{\pi}} - \int_{-\pi}^{0}\left(\frac{-\cos nx}{n}\right)dx\right) + \frac{2}{\pi}\left(\frac{2x\cancel{\pi}}{\cancel{\pi}} - \int_{0}^{\pi}\frac{\cos nx\,dx}{n}\right)$$

$$\Rightarrow b_n = \frac{1}{\pi}\cdot -2x - \left[\cancel{\frac{-\sin nx}{n^2}}\Big|_{-\pi}^{0}\right] + \frac{4x}{\pi} - \left(\cancel{\frac{-\sin nx}{n^2}}\Big|_{0}^{\pi}\right)$$

$$\Rightarrow b_n = -\frac{2x}{\pi} + \frac{4}{\pi}x$$

$$\boxed{b_n = \frac{2x}{\pi}}$$

Now putting values in eq (i)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\boxed{f(x) = \frac{3}{4}\pi + \sum_{n=1}^{\infty} \frac{-2}{n^2\pi} \cos nx + \sum_{n=1}^{\infty} \frac{2x}{\pi} \sin nx}$$

Hence it is the required series.

QUESTION No 03

Solve the initial value problem

$$y'' - 4y + 13y = 8 \sin 3x$$

$$y(0) = 1 \quad \& \quad y'(0) = 2$$

SOLUTION :-

Associated homogeneous Eq of ① is

$$y' - 4y' + 13y = 8 \sin 3x$$

So

$$\frac{d^2y}{dx^2} - 4\frac{d}{dx}y + 13y = 8 \sin 3x$$

$$y\left(\frac{d^2}{dx^2} - 4\frac{d}{dx} + 13\right) = 8 \sin 3x$$

Now, Put $\frac{d}{dx} = D$

$$y(D^2 - 4D + 13) = 8 \sin 3x$$

Put $D = \Delta$

$y(\Delta^2 - 4\Delta + 13) = 8 \sin x$

Now Characteristic equation

$\Delta^2 1 - 4\Delta + 13$

$a = 1 \;,\; b = -4 \;,\; C = 13$

Now from Quadratic equati

$\Rightarrow \Delta = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1 \times 13)}}{2(1)}$

$\Rightarrow \Delta = \dfrac{4 \pm \sqrt{16 - 52}}{2}$

$\Rightarrow \Delta = \dfrac{4 \pm \sqrt{-36}}{2}$

$\Rightarrow \Delta = \dfrac{4 \pm \sqrt{36}\, i}{2}$

$\Rightarrow \Delta = \dfrac{4 \pm 6i}{2} \Rightarrow \dfrac{\cancel{2}(2 \pm 3i)}{\cancel{2}}$

$\Rightarrow D = 2 \pm 3i$ ∴ ~~after taking~~

Now $m_1 = 2 + 3i$
$m_2 = 2 - 3i$

Now

$$y_c = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) \quad -(A)$$

Diff w.r.t x

$$y_p' = -3A \sin 3x + 3B \cos 3x$$

again diff w.r.t x

$$y_p'' = -9A \cos 3x - 9B \sin 3x$$

Putting in eq :

$$= (-9A \cos 3x - 9B \sin 3x) - 4(-3A \sin 3x + 3B \cos 3x) + B(A \cos 3x + B \sin 3x) - B \sin 3x$$

$$= -9A \cos 3x - 12B \cos 3x + AB \cos 3x - 9B \sin 3x + 12A \sin 3x + 13B \sin 3x - 8 \sin 3x$$

$$\Rightarrow (-9A - 12B + 13A) \cos 3x + (-9B + 12A + 13B) \sin 3x = 8 \sin 3x$$

Comparing co-efficient

$$\sin 3x = 4B + 12A = 8 \quad -(a)$$

$$4A - 12B = 0 \Rightarrow 4A - 12B$$

$$\boxed{A = 3B} \quad -(b)$$

put b In a

$4B + 12(3B) = 8$

$40B = 8$

$\boxed{B = 1/5}$ — (i)

put (i) In (b)

$\boxed{A = 3/5}$ — (d)

put c ~~&~~ d in (x)

$y_p = 3/5 \cos 3x + 1/5 \sin 3x$ — (B)

the general solution is ;

$y = y_c + y_p$

$y = e^{2x}(4\cos 3x + C_2 \sin 3x) + 3/5 \cos 3x + 1/5 \sin 3x$ — (E)

Now we need to find the values of C_1 & C_2 for this

Put $x = 0$ & $y = 1$ in (C)

$$1 = e^{2(0)}\left(C_1 \cos 3(0)\right) + \left(2 \sin 3(0)\right) + \frac{3}{5} \cos 3(0) + \frac{1}{2} \sin 3(0)$$

$$1 = (C_1) + 2(0) + \frac{3}{5}(1) + \frac{1}{5}(0)$$

$$1 = C_1 + \frac{3}{5}$$

$$\boxed{C_1 = \frac{2}{5}} \quad \text{— (a')}$$

Differentiate C w.r.t 'x'

$$y' = C_1\left(2e^{2x} \cos 3x - 3e^{2x} \sin 3x\right) + C_2\left(2e^{2x} \sin 3x + 3e^{2x} \cos 3x\right) - \frac{6}{5}\left(\sin 3x + \frac{3}{5} \cos 3x\right. \quad \text{— (D)}$$

Put $y' = 2$, $x = 0$ in (D)

$$y' = C_1\left(2e^{2x} \cos 3x - 3e^{2x} \sin 3x\right) + C_2\left(2e^{2x} \sin 3x + 3e^{2x} \cos 3x\right. - \frac{6}{5} \sin 3x + \frac{3}{5} \cos 3x$$

Put $y' = 2$ & $x = 0$

$$2 = C_1\left(2e^{2(0)} \cos 3(0) - 3e^{2(0)} \sin 3(0)\right) + C_2\left(2e^{2(0)} \sin(0) + 3e^{2(0)} \cos 3(0)\right)$$

$$2 = C_1(2) + C_2(3) - 0 + \frac{3}{5}$$

$$2 = 2C_1 + 3C_2 + \frac{3}{5}$$

Put $C_1 = \frac{2}{5}$

$$2 = 4/5 + C_2 + 3/5$$
$$2 = 7/5 + 3C_2$$

$$3C_2 = 3/5$$
$$\boxed{C_2 = 3/15} \quad \text{— } a''$$

Put a' & a'' in (c)

$$y = e^{3x}\left(2/5 \cos 3x + 3/15 \sin 3x\right) + 3/5 \cos 3x + 1/5 \sin 3x$$

$$\boxed{y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{15}\cos 3x + \frac{1}{5}(\sin 3x)}$$

Hence this is the required equation

QUESTION NO 04

$$D^2 - (DD')z = \cos x \cos 2y$$

SOLUTION :-

$$(D^2 - DD')z = \cos x \cos 2y$$

the given PDE can be rewrite as $D(D-D')z = \cos x \cos 2y$ in CF is given by

$$CF = \phi_1(y) + \phi_2(y+x)$$

while it's PI is given by

$$PI = \frac{1}{(D^2-DD')}, \frac{1}{2}\left[\cos(x+2y) + \cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(-1+2)} \cos(x+2y) + \frac{1}{(-1-2)} \cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(-1+2)} \cos(x+2y) + \frac{1}{(-1+2)} \cos(x-2y)\right]$$

$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Hence the complete solution of PDE is

$$z = \phi_1 - (y) + \phi_2(y+x) + \frac{1}{2}\cos(x+2y) + \frac{1}{6}\cos(x-y)$$