

Name

Muhammad Ali Khan Khattak

ID

7918

Subject

Differential equations

Section

(A)

Semester

Summer

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(1)

Question # 1

Find the Fourier series representation of $f(t) = 1+t, -\pi \leq t \leq \pi$.

Answer # 1

Solution :-

$$f(t) = 1+t \quad -\pi \leq t \leq \pi$$

Here we use the formula:-

Formula :-

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos t + \sum_{n=1}^{\infty} b_n \sin t \quad \text{--- eq(1)}$$

So,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+t) dt$$

$$a_0 = \frac{1}{2\pi} \left[t + \frac{t^2}{2} \right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi} \left(\pi - (-\pi) + \frac{\pi^2}{2} - \left(-\frac{\pi^2}{2} \right) \right)$$

$$a_0 = \frac{1}{2\pi} \left(2\pi + \frac{2\pi^2}{2} \right)$$

$$a_0 = \frac{1}{2\pi} (2\pi + \pi^2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) (\cos nt) dt \quad (2)$$

$$a_n = \frac{1}{\pi} \left((1+t) \frac{\sin nt}{n} \Big|_{-\pi}^{\pi} - \int \left(\frac{\sin nt}{n} dt (1+t) \right) \right)$$

$$a_n = \frac{1}{\pi} \left(\left((1+t) \frac{\sin nt}{n} - \frac{\cos nt}{n^2} \right) \Big|_{-\pi}^{\pi} \right)$$

$$a_n = \frac{-1}{n^2 \pi} (\cos n \pi - \cos n (-\pi))$$

$$a_n = \frac{-1}{n^2 \pi} (-1 - (-1))$$

$$a_n = 0$$

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$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) \sin nt \, dt$$

$$b_n = \frac{1}{\pi} \left((1+t) \int_{-\pi}^{\pi} \sin nt - \int \left(\int \sin nt - \frac{d}{dt} (1+t) dt \right) \right)$$

$$b_n = \frac{1}{\pi} \left(\frac{(1+t)(-\cos nt)}{n} \Big|_{-\pi}^{\pi} - \int \left(\frac{-\cos nt}{n} (1) \right) \right)$$

$$b_n = \frac{1}{\pi} \left(\frac{-(1+t)(\cos nt)}{n} \Big|_{-\pi}^{\pi} + \left(\frac{\sin nx}{n^2} \Big|_{-\pi}^{\pi} \right) \right)$$

$$b_n = \frac{-1}{n\pi} \left((1+\pi)(\cos n\pi) - \left((1+(-\pi)) (\cos n\pi) \right) \right)$$

$$b_n = \frac{-1}{n\pi} (\cos n\pi + \pi \cos n\pi - \cos n\pi + \pi \cos n\pi)$$

$$b_n = \frac{-1}{n\pi} (2\pi \cos n\pi)$$

Here,

$$\cos n\pi = \frac{(-1)^{n+1}}{n}$$

$$b_n = \frac{2}{n} (-1)^{n+1} \quad (4)$$

So eqn become

$$f(x) = \frac{1}{2\pi} (2\pi + \pi^2) + 0 + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin t$$

Question #2

Calculate the characteristic equation the Eigen values of the system, where A is given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix}$$

Answer #2

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix}$$

Eigen Values = ?

Solution:-

Step = 1

we have;

$$(A - \lambda I)x = 0 \quad \because A = \text{given Matrix}$$

Step = 2

we have;

The characteristic equation is given by;

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 1-\lambda & 4 \\ 0 & 2 & 2-\lambda \end{vmatrix} = 0$$

Step = 3;

$$\lambda^3 - \left| \begin{array}{c} \text{sum of} \\ \text{diagonal elem} \end{array} \right| \lambda^2 + \left| \begin{array}{c} \text{sum of} \\ \text{diagonal} \\ \text{minors} \end{array} \right| \lambda - |A| = 0 \quad \text{--- (B)}$$

$$\text{Sum of diagonal elements} = 1 + 1 + 2 = 4$$

$$\text{Sum of diagonal minors} = \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$$

$$= (-6) + (2) + (1)$$

$$= -6 + 2 + 1$$

$$= -3$$

By putting values in eq(B),

$$\lambda^3 - 4\lambda^2 - 3\lambda - |A| = 0 \quad \text{--- (C)}$$

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 1(2-8) - 0 + 1(6-0)$$

$$= -6 + 6$$

$$= 0$$

⑦

By putting values in eq(c);

$$\lambda^3 - 4\lambda^2 - 3\lambda - 0 = 0$$

$$\lambda^3 - 4\lambda^2 - 3\lambda = 0$$

$$\lambda(\lambda^2 - 4\lambda - 3) = 0$$

$$\lambda = 0$$

$$\lambda^2 - 4\lambda - 3 = 0$$

Using quadratic formula;

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

∴ a = 1
b = -4
c = -3

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 + 12}}{2} = \frac{4 \pm \sqrt{28}}{2}$$

$$\lambda = \frac{4 + \sqrt{28}}{2}, \lambda = \frac{4 - \sqrt{28}}{2}$$

we have eigen values;

~~$$\lambda = \left(0, \frac{4 + \sqrt{28}}{2}, \frac{4 - \sqrt{28}}{2} \right)$$~~

$$\lambda = \left(0, \frac{4 + \sqrt{28}}{2}, \frac{4 - \sqrt{28}}{2} \right)$$
 required solution.

Question #3

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Solve the following system of linear equations

$$5x + 4z + 2m = 3$$

$$x - y + 2z + m = 1$$

$$4x + y + 2z = 1$$

$$x + y + 2z + m = 0$$

Answer #3

Solution :-

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right] \xrightarrow{R_4 R_2}$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & -1 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1+4/5 & 1 \\ 0 & -1 & +6/5 & +4/5 & 3/5 \\ 0 & 2 & -1 & 0 & -1 \end{array} \right] \xrightarrow{-1/5 \times R_3}$$

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$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -1 & 6/5 & 4/5 & 3/5 \\ 0 & 0 & 7/5 & 8/5 & 1/5 \end{array} \right]$$

$\underbrace{5 \times R_3}$ and $\underbrace{5 \times R_4}$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 6 & 1 \end{array} \right]$$

$\underbrace{5R_3}$ and $\underbrace{5R_4}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 2/5 & 3/5 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right]$$

$\frac{1}{5} \times R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -1 & 6/5 & 1/5 & 2/5 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right]$$

$\underbrace{R_2 \times 5}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right]$$

$\underbrace{R_3 - R_2}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 8/5 & 1/7 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$\underbrace{R_3 \leftrightarrow R_4}$
 $\underbrace{1/7 \times R_3}$
 $\underbrace{1/3 \times R_4}$

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$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\underbrace{C_2 \times -5}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 6 & 1 & 2 \\ 0 & 0 & 1 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & -5 & 26/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1/2 & 3/4 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\underbrace{5/4 \times R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 126/84 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

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$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & -\frac{11}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & -\frac{11}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{array} \right]$$

$$(x, y, z, m) = \left(\frac{3}{4}, \frac{3}{2}, -\frac{11}{2}, \frac{1}{3} \right)$$

$$x = \frac{3}{4}$$

$$y = \frac{3}{2}$$

$$z = -\frac{11}{2}$$

$$m = \frac{1}{3}$$

Question # 4

(12)

Verify that

$$u(x, t) = \sin(x + 2t)$$

is a

solution

of 1-D wave equation

Answer # 4

Solution :-

Given that

$$u(x, t) = \sin(x + 2t)$$

Differentiate w.r.t x partially

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (\sin(x + 2t))$$

$$\frac{\partial u}{\partial x} = \cos(x + 2t) \frac{\partial}{\partial x} (x + 2t)$$

$$\frac{\partial u}{\partial x} = \cos(x + 2t) (1 + 0)$$

$$\therefore \frac{\partial u}{\partial x} = \cos(x + 2t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \cos(x + 2t)$$

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$$\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t) \frac{\partial}{\partial x} (x+2t)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t)(1+0)$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t)}$$

and

$$u(x,t) = \sin(x+2t)$$

Differentiate w.r.t "t"

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \sin(x+2t)$$

$$\frac{\partial u}{\partial t} = \cos(x+2t) (0+2)$$

$$\frac{\partial u}{\partial t} = 2\cos(x+2t)$$

$$\frac{\partial^2 u}{\partial t^2} = (2) - \sin(x+2t) (0+2)$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = -4\sin(x+2t)}$$

As we know that one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$-4 \sin(x+2t) = c^2 [-\sin(x+2t)]$$

$$-4 \sin(x+2t) = -c^2 \sin(x+2t)$$

$$-4 \sin(x+2t) + c^2 \sin(x+2t) = 0$$

For the arbitrary constant $c = \pm 2$

$$-4 \sin(x+2t) + (\pm 2)^2 \sin(x+2t) = 0$$

$$-4 \sin(x+2t) + 4 \sin(x+2t) = 0$$

$$0 = 0$$

Then it will be verified for the arbitrary constant

$$c = 2$$