Name = Shahid-Nasir.
ID # 15531.
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Teacher = Sir Latif Jan.

Q1(a):- Define differential equation along with 2 examples?

Ans(a):- **Differential equation:**

A differential equation is an equation that relates one or more functions and their derivatives.

Example#1:
Find the Solution of $y'=5$ whose $x=0, y=2$.

Ans $y'=5$

$dy = 5\,dx$.

integrating B.S

$y = 5x + K$

Applying the boundary condition: $x=0, y=2$ $\therefore K=2$.

$$\boxed{y = 5x + 2}$$

Example # 1

Find the particular solution of $y''' = 0$.

where $y(0) = 3$, $y'(1) = 4$, $y''(2) = 6$.

Since $y''' = 0$

$y'' = A$ ∵ where A is constant

integrating again gives.

$y' = Ax + B$ ∵ B is also constant

integrating once again.

$y = \frac{Ax^2}{2} + Bx + C$ ∵ C is constant

So,

$y(0) = 3$, $y'(1) = 4$, $y''(2) = 6$

Now, $y(0) = 3$ gives $C = 3$

$y''(2) = 6$ gives $A = 6$.

$y'(1) = 4$ gives $B = -2$

$y = 3x^2 - 2x + 3$

Checking the solution by differentiating & substituting:

$y' = 6x - 2$

$\Rightarrow y'(1) = 6(1) - 2 = 4$.

$y'' = 6$.

$y''' = 0$ Ans

b):- Define a separable differential equation?

Ans:- **Separable D.E:**

A first order differential equation $y' = f(x, y)$ is called a separable equation. If the function $f(x, y)$ can be factored in to the product of two functions of x & y:

$$f(x, y) = p(x)\, h(y),$$

where $p(x)$ & $h(y)$ are continuous function.

Ans

i):- Solve the following IVP using separable D.E & find the interval of validity of the solution.

a):- $y' = \dfrac{xy^3}{\sqrt{1+x^2}}$ $\qquad y(0) = -1.$

Solution

$$y^{-3}\, dy = x(1+x^2)^{-\frac{1}{2}}\, dx$$

$$\int y^{-3}\, dy = \int x(1+x^2)^{-\frac{1}{2}}\, dx$$

$$\frac{1}{-2y^2} = \sqrt{1+x^2} + C$$

$$-\frac{1}{2} = \sqrt{1} + C$$

$$C = -\frac{3}{2}$$

$$\frac{1}{-2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

$$y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Now finding interval of validity.

$$3 - 2\sqrt{1+x^2} > 0.$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4(1+x^2)$$

$$\frac{9}{4} > 1 + x^2$$

$$\frac{5}{4} > x^2.$$

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

$x = 2$. Interval of validity.

Ans

d). $y' = e^{-y}(2x-4)$ $y(5) = 0$.

Soln

Step #1: Multiplying by e^y & by dx.

$$e^y \, dy = (2x-4) \, dx$$

$$\int e^y \, dy = \int (2x-4) \, dx$$

$$e^y = x^2 + 4x + C$$

Natural log

$$y = \ln(x^2 + 4x + C)$$

finding C

$$y(5) = \ln((5)^2 - 4(5) + C)$$

$$\ln(5 + C) = 0.$$

$$5 + C = 1.$$

$$C = -4.$$

$$y = \ln(x^2 - 4x - 4.$$

Ans

Day: M T W T F S

Q2:- a) Solve the following IVP using linear D.E method.

(i):- Explain the steps for solving linear D.E.

Ans:- Substitute $y = UV$
Factor the parts involving V.
Put the V term equal to zero.
Solve using separation of variables to find U.
Substitute U back in to the equation we got at step 2.
Solve that to find V.
Finally, substitute U & V in to $y = UV$ to get our solution.

ii):- $\cos(x)y' + \sin(x)y = 2\cos^2(x)\sin(x) - 1$ $\quad y(\frac{\pi}{4}) = 3\sqrt{2}$

$0 \le x \le \frac{\pi}{2}$.

Ans:-

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos x}$$

$$y' + \tan(x)y = 2\cos^2(x)\sin x - \sec(x).$$

$$\mu(t) = e^{\int \tan(x)\,dx} = e^{\ln|\sec(x)|} = e^{\ln \sec(x)}$$

$$\sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$(\sec(x)y)' = 2\cos(x)\sin(x) - \sec^2(x)$$

$\int$ B.S.

$$\int (\sec(x)y(x))'\,dx = \int 2\cos(x)\sin(x) - \sec^2(x)$$

$$\sec(x)y(x) = \int \sin(2x) - \sec^2(x)\,dx$$

$$\sec(x)y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + C$$

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + C$$

$$= -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + C\cos(x)$$

Put the value of "y" & "x"

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin$$

$+ \cos$

$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + C\frac{\sqrt{2}}{2}$$

$$C = 7$$

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x).$$

Ans

Q3:- Solve the following IVP for ExE & find the interval of validity for the solution.

i) 1- $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$, $y(0) = -3$.

Ans $M = 2xy - 9x^2$, $M_y = 2x$

$N = 2y + x^2$, $N_x = 2x$

Now, how do we actually find $\psi(x,y)$?

$\psi_x = M$

$\psi_y = N$.

$\psi = \int M dx$ or $\psi = \int N dy$.

$\psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$.

$h'(y) = 2y + 1$.

$h(y) = \int 2(y) + 1 \, dy = y^2 + y + K$.

$\psi(x,y) = x^2 y - 3x^2 + y^2 + y + K = y^2 + (x^2+1)y - 3 + K$.

$y^2 + (x^2+1)y + 3x^2 + K = C$.

$y^2 + (x^2+1)y - 3x^2 = C - K$

$y^2 + (x^2+1)y - 3x^2 = C$.

Initial Condition to find C

$(-3)^2 + (0+1)(-3) - 3(0)^3 = C$ $\therefore C = 6$

Put the value of C.

$y^2 + (x^2+1)y - 3x^2 - 6 = 0$.

Using Quadratic formula

$y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^2-6)}}{2(1)}$.

$= \frac{-(x^2+1) \pm \sqrt{x^2 + 12x^3 + 2x^2 + 25}}{2}$

Pg # 10.

$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} \neq -3, 2$

$y(x) = \frac{-(x^2+1) - \sqrt{x^4 + 12x^2 + 25}}{2}$

$x^4 + 12x^3 + 2x^2 + 25 = 0.$

ii $\dfrac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0$ Pg # 11

$y(5) = 0$

Soln

$M = \dfrac{2ty}{t^2+1} - 2t \qquad M_y = \dfrac{2t}{t^2+1}$

$N = \ln(t^2+1) - 2 \qquad N_t = \dfrac{2t}{t^2+1}.$

$\int$ first one.

$\Psi(x,y) = \int \dfrac{2ty}{t^2+1} - 2t \; dy = y\ln(t^2+1) - t + h(y)$

Now differentiate.

$\Psi_y = \ln(t^2+1) + h'(y) \quad \ln(t^2+1) - 2.$

$h'(y) = -2 \Rightarrow h(y) = -2y.$

$\Psi(t,y) = y\ln(t^2+1) - t^2 - 2y.$

$y\ln(t^2+1) - t^2 - 2y = C.$

$C = -25$

$y(\ln(t^2+1) - 2) - t^2 = -25.$

$y(t) = \dfrac{t^2 - 25}{\ln(t^2+1) - 2}$

$\ln(t^2+1) - 2 = 0.$

$\ln(t^2+1) = 2.$

$t^2 + 1 = e^2$

$t = \pm\sqrt{e^2 - 1}$ Ans