Name = Nadeem Iqbal

ID = 7277

Sa assignment = differnial equation

Submited to : Shomila mazhar

Cauchy - Euler ODE's

QNo.1 $x^3 y''' + 2x^2 y'' + 2y = 10x + 10/x$ (1)

Sol:

Put $x = e^t$ then

$\frac{dx}{dt} = e^t \Rightarrow dt/dx = e^{-t}$

Now $dy/dx = dy/dt \cdot dt/dx = dy/dt \cdot e^{-t}$

or $y' = dy/dx = e^{-t} Dy \quad \because d/dt \to D$

Similarly

$y'' = e^{-2t} [D(D-1)]y$

$y''' = e^{-3t} [D(D-1)(D-2)]y$

using this values in (1)

$e^{3t} \cdot e^{-3t} [D(D-1)(D-2)]y + 2e^{2t} \cdot e^{-2t} [D(D-1)]y + 2y = 10e^t + 10e^{-t}$

$(D^3 - 3D^2 + 2D)y + (2D^2 - 2D)y + 2y = 10e^t + 10e^{-t}$

$D^3 y - D^2 y + 2y = 10e^t + 10e^{-t}$

$\frac{d^3 y}{dt^3} - \frac{d^2 y}{dt^2} + 2y = 10e^t + 10e^{-t}$ (2)

the associated homogenous equation of (1) is

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 0$$

$$d/dt^2 = K^2, \quad d^3/dy^3 = K^3$$

$$\cancel{K^3y - K^2y}$$

$$(K^3y - K^2y + 2y) = 0$$

$$(K^3 - K^2 + 2)y = 0$$

For non-trivial sol $y \neq 0$

$$K^3 - K^2 + 2 = 0$$

Root are

$$K = -1, \ 1 \pm i$$

$$\boxed{y_c(t) = Ae^{-t} + (B\cos t + C\sin t)e^{t}}$$

Which is Complementary Solution.

QNo.5 $x^2y'' + 2xy' - 6y = 10x^2$ $\quad y(1) = 1 \quad y'(1) = -6$

Sol let $x = e^t$ so that $\log x = t$
being Cauchy euler equation

$$[D(D-1) + 2D - 6]y = 10e^{2t}$$

$$[D^2 - D + 2D - 6]y = 10e^{2t}$$

$$[D^2 + D - 6]y = 10e^{t} \quad \text{—— (1)}$$

the Characterstic Equation is

$$D^2 + D - 6 = 0$$

$$D^2 + 3D - 2D - 6 = 0$$

$$(D+3)(D-2) = 0$$

$$D = 2, -3$$

So its Complementar Function

$$y_c = C.F = C_1e^{-3t} + C_2e^{2t}$$

Now the particular integral is

$$y_P = P.I = \frac{1}{D^2 + D - 6} 10e^{2t}$$

$$= 10 \frac{1}{(2)^2 + 2 - 6} e^{2t} \Rightarrow 10 \frac{1}{0} e^{2t} \text{ (Case fail)}$$

$$= 10 \frac{t}{2D+1} e^{2t} \Rightarrow 10 \frac{t}{4+1} e^{2t}$$

$$= \frac{10}{5} te^{2t} \Rightarrow 2te^{3t}$$

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Hence general equation

$y = y_c + y_p$

$= c_1 e^{-3t} + c_2 e^{2t} + 2te^{2t} \quad \because x = e^t$

$y = c_1 x^{-3} + c_2 x^2 + 2(\log x)x^2$ — (i)

Now put $y(1) = 1$ ie $x = 1$ $y = 1$ in (iii)

$1 = c_1 + c_2 + 2\log = c_1 + c_2$ — (iv)

Now differentiate ~~w.r.t~~ (ii) w.r.t. x

$y' = 3c_1 x^{-4} + 2c_2 x + y x \log x + 2x$

Now put $y'(1) = -6$ i.e $x = 1$ and $y' = -6$

$-6 = -3c_1 + 2c_2 + 0 = -3c_1 + 2c_2 + 2 = -6$

$\Rightarrow -3c_1 + 2c_2 = -8 \Rightarrow$ (x)

So multiplying (iv) by 2 and subtracting ~~by~~ from eq (x)

$$2c_1 + 2c_2 = 2$$
$$\underline{\mp 3c_1 \pm 2c_2 = \mp 8}$$
$$5c_1 = 10$$

$\boxed{c_1 = 2}$ $\boxed{c_2 = -1}$

Put the value of c_1, c_2 in eq (ii)

$y = 2x^{-3} - x^2 + 2\log x (x^2)$

or

$$\boxed{y = \frac{2}{x^3} - x^2 + 2x^2 \log x}$$

Q4: $x^2y'' + 7xy' + 5y = x^5$ $y(0) = 2$ $y(1) = 2$

Sol Let $x = e^t$ $t = \log x$. $D = d/dt$

Now $xy' = Dy \Rightarrow x^2y'' = D(D-1)y$

then $(D(D-1) + 7D + 5)y = e^{5t}$

$(D^2 - D + 7D + 5)y = e^{5t}$

$(D^2 + 6D + 5)y = e^{5t}$

Char eq is

$D^2 + 6D + 5 = 0$

$D^2 + 5D + D + 5 = 0$

$D = -5, -1$

Complementary equation is

$C.F = C_1 e^{-5t} + C_2 e^{-t}$

P integral

$P.I = \frac{1}{D^2 + 6D + 5} e^{5t}$

$\frac{1}{5^2 + 6(5) + 5} e^{5t}$ replacing D by 5

$= \frac{1}{60} e^{5t}$

then $y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t}$

$y = C_1 x^{-5} + C_2 x^{-1} + \frac{1}{60} x^5$

$y' = -5C_1 x^{-6} - C_2 x^{-2} + \frac{1}{12} x^4$

$y(0) = 2$ $x = 0, y = 2$

$2 = C_1 + C_2 + \frac{1}{60}$

$C_1 + C_2 = 119/60$ — (A)

$y'(1) = 2 \quad x = 1 \quad y' = 2$

$2 = -5c_1 - c_2 + 1/12$

$-5c_1 - c_2 = \frac{23}{12}$ — B

A + B

$-4c_1 = \frac{234}{60} \Rightarrow c_1 = \frac{-117}{120}$

now $y = \frac{-117}{120} x^{-5} + c_2 x^{-1} + \frac{1}{60} x^5$

$c_1 = \frac{-117}{120}$ Put in A $\frac{-117}{120} + c_2 = \frac{119}{60}$

$c_2 = \frac{119}{60} + \frac{117}{120}$

$\frac{238}{120} + 117 = \frac{355}{120}$

Q5. $(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$

Sol

Let $x+1 = e^t \Rightarrow x = e^t - 1$

differentiate $(\log x+1) = t$

Also $(x+1)y' = Dy_1$ $(d/dt = D$ and $D = d/dx)$

$\cancel{x+D}$

$(x+1)^2 y'' = D(D-1)y$

then eq(1) $\Rightarrow$ $(D(D-1) - 3D + 4)y = (e^t - 1)^2$

$(D^2 - 4D + 4)y = e^{2t} - 2e^t + 1)$

Character eq is $D^2 - 4D + 4 = 0$

$(D-2)^2 = 0$

$D = 2, 2$

the Complementary function is

$C.F = (C_1 + C_2 t)e^{2t}$

Also particular integral is

$P.I \; \frac{1}{(D-2)^2}(e^{2t} - 2e^t + 1)$

$= \frac{1}{(D-2)^2}e^{2t} - 2\frac{1}{(D-2)^2}e^t + \frac{1}{(D-2)^2}$ —(2)

NOW

$\frac{1}{(D-2)^2}e^{2t} = \frac{1}{g(D-2)^2}e^t = \cancel{\;} \frac{1}{0}e^{2t}$ (case fail)

$\frac{1}{(D-2)^2}e^{2t} = \frac{1}{2(1-2)^2}e^t = \frac{t^2 e^t}{2}$

and $\quad 2\frac{1}{(D-2)^2}e^{t} = 2\frac{1}{(1-2)^2}e^{t} = 2e^{t}$

and $\quad \frac{1}{(D-2)^2}e^{0(t)} = \frac{1}{(D-2)^2}e^{0} = 1/4$

eq ② $\quad P.I = 1/2\, t^2 e^{2t} - 2e^{t} + 1/4$

Hence complete solution is

$y = C.F + P.I$

$y = (C_1 + C_2 t)e^{2t} + 1/2\, t^2 e^{2t} - 2e^{t} + 1/4$

~~Put value of~~

$y = C_1 + C_2 \log(x+1)\ \cancel{x+1)}\ (x+1)^2 +$

$1/2\{(\log x+1)^2 (x+1)^2\} - 2(x+1) + 1/4$

or

$y = C_1 + C_2 \log(x+1)\ (x+1)^2 +$

$\boxed{1/2\{(\log(x+1))^2 (x+1)^2\} - 2x - 7/4}$

Which is the required

(4)

Q.No.2 $x^3y''' + 4x^2y'' - 5xy' - 15y = x^4$

Sol: $x^3y''' + 4x^2y'' - 5xy' - 15y = x^4$ — (1)

Put $x = e^t$ then $dy/dx = y' = e^t\ Dy$; $D = d/dt$

and $\frac{d^2y}{dx^2} = y'' = e^t\ (D(D-1))y$

Using these values in (1)

$D(D-1)y + 2De^t\ e^0y - 6y = 10e^t$

$D^2y - Dy + 2Dy - 6y = 10e^{2t}$

$d^2y/dt^2 + dy/dt - 6y = 10e^{2t}$

$d^2y/dt^2 + dy/dt - 6y = 10e^{2t}$ — (2)

Associated

$d^2y/d + dy/dt - 6y = 0$ — (3)

d/dt in $d^2/dt^2 = m^2$ then (3)

$m^2y + my - 6y = 0 \Rightarrow (m^2 + m - 6)y = 0$

roots are $m = -3, 2$

The Complementary Solution

$y_c = C_1e^{-3t} + C_2e^{2t}$ — (4)

Now find W, W_1, W_2

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-3t} & e^{2t} \\ -3e^{3t} & 2e^{3t} \end{vmatrix} = 2e^{-t} + 3e^{-t} = 5e^{-t}$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{2t} \\ 10e^{2t} & 2e^{2t} \end{vmatrix} = -10e^{4t}$

$$W_2 \begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix} = \begin{vmatrix} e^{-3t} & 0 \\ -3e^{-3t} & 10e^{2t} \end{vmatrix} = 10e^{-t}$$

Now $u_1' = \int \frac{W_1}{W_2}\,dt = \int \frac{-10^{4t}}{5e^{-t}}\,dt = -2\int e^{5t}\,dt = \frac{-2}{5}e^{5t}$