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Subject Differential equation

Date 20/8/2020

Q1. $\frac{dy}{dt} = e^{y-t} \sec(y)(1+t^2)$ $\quad y(0)=0$

Solution:-

$$\frac{dy}{dt} = e^{y-t} \sec(y)(1+t^2)$$

$y(0) = 0$ so $x = 0$ $y = 0$

$$dy = e^{y} . e^{-t} \sec(y) (1+t^2) dt$$

$$\frac{1}{e^{y} . \sec(y)} dy = (1+t^2) e^{-t} dt$$

As $\cos(y) = \frac{1}{\sec(y)}$

$$\int e^{-y} \cos y dy = \int (1+t^2) e^{-t} dt$$

Using integration by parts

$$e^{-y} \int \cos y dx - \int \left(\int \cos y . \frac{d}{dy} e^{-y} \right) = (1+t^2) \int e^{-t}$$

$$- \int \left(\int e^{-t} . \frac{d}{dt} (1+t^2) \right) \quad eq \to 1$$

L.H.S

$$e^{-y} \int \cos y dx - \int \left(\int \cos y . \frac{d}{dy} e^{-y} \right)$$

$e^{-y}\sin y - \int(\sin y . e^{-y}(-1))$

$e^{-y}\sin y + \int(\sin y . e^{-y})$

$e^{-y}\sin y + \int(e^{-y}\sin y)$

Again using integration by parts

$e^{-y}\sin y + e^{-y}(-\cos y) - \int(\int \sin y \frac{d}{dy} e^{-y})$

$e^{-y}\sin y + e^{-y}(-\cos y) - \int(-\cos y \frac{e^{-y}}{-1})$

$e^{-y}\sin y - e^{-y}\cos y - \int(\cos y\, e^{-y})$

since $\int(\cos y e^{-y}) = L.H.S$

Since it is again same to the first one so L.H.S will become.

$L.H.S = e^{-y}(\sin y - \cos y) - L.H.S$

$2 L.H.S = e^{-y}(\sin y - \cos y)$

$L.H.S = \frac{e^{-y}(\sin y - \cos y)}{2}$

Now taking R.H.S

$$\int (1+t^2)\, e^{-t}\, dt$$

$$= (1+t^2)\int e^{-t} - \int\left(\int e^{-t} \frac{d}{dt}(1+t^2)\right)$$

$$= (1+t^2)\, e^{-t} - \int\left(-e^{-t}(2t)\right)$$

$$= (1+t^2)e^{-t} + \int\left((2t)e^{-t}\right)$$

Again using integration by parts.

$$= (1+t^2)e^{-t} + \left(2t\int e^{-t} - \int\left(\int e^{-t} \frac{d}{dt} 2t\right)\right)$$

$$= (1+t^2)\, e^{-t} + \left(-2te^{-t} - \int\left(-e^{-t}2\right)\right)$$

$$= (1+t^2)\, e^{-t} + \left(-2te^{-t} + \int\left(-e^{-t}2\right)\right)$$

$$= (1+t^2)e^{-t} + \left(-2te^{-t} - 2e^{-t}\right) + c$$

$$= (1+t^2)e^{-t} - 2te^{-t} - 2e^{-t} + c$$

$$= e^{-t} - e^{-t}t^2 - 2te^{-t} - 2e^{-t} + c$$

$$= -(t^2+2t+3)\, e^{-t} + c \quad = \quad R.H.S$$

Now take L.H.S = R.H.S

$$\frac{e^{-y}(\sin y - \cos y)}{2} = -(t^2+2t+3)\,e^{-t} + c$$

We know that

$t = 0 \qquad y = 0$

Put it above

$$\frac{1}{2}(0-1) = -3 + c$$

$$c = 5/2$$

Put value of c

$$\boxed{\frac{e^{-y}}{2}(\sin y - \cos y) = -(x^2+2t+3)\,e^{-t} + 5/2}$$

Ans

Q2. $\left(\sqrt{x+y}+\sqrt{x-y}\right)dx-\left(\sqrt{x+y}-\sqrt{x-y}\right)dy=0$

Solution:

$$\left(\sqrt{x+y}+\sqrt{x-y}\right)dx-\left(\sqrt{x+y}-\sqrt{x-y}\right)dy=0$$

$$\frac{dy}{dx}=\frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}} \rightarrow (1)$$

This is Homogeneous Differential eq in x and y to solve this

Put $y=vx$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Thus eq (1) becomes

$$v+x\frac{dv}{dx}=\frac{\sqrt{x+vx}+\sqrt{x-vx}}{\sqrt{x+vx}-\sqrt{x-vx}}$$

$$v+x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}$$

$$v+x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}\times\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}+\sqrt{1-v}}$$

$$v+x\frac{dv}{dx}=\frac{1+\cancel{v}+1-\cancel{v}+2\sqrt{1-v^2}}{2v}$$

$$v+x\frac{dv}{dx} = \frac{\cancel{2}(1+\sqrt{1-v^2})}{\cancel{2}v}$$

$$v+x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}}{v}.$$

$$x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{\sqrt{1-v^2}(1+\sqrt{1-v^2})}{v}$$

$$\frac{v\,dv}{\sqrt{1-v^2}(1+\sqrt{1-v^2})} = \frac{dx}{x}$$

Taking integrals on b/s

$$\int\frac{v\,dv}{\sqrt{1-v^2}(1+\sqrt{1-v^2})} = \int\frac{dx}{x}$$

put $1+\sqrt{1-v^2}=t$

$$\Rightarrow \frac{1}{2}(1-v^2)^{-1/2}(-2v)\cancel{dv}=dt$$

$$\frac{v\,dv}{\sqrt{1-v^2}} = -dt$$

$$\int-\frac{dt}{t} = \int\frac{dx}{x}$$

$$-\ln t = \ln x + \ln c$$

$$-\ln\left(1+\sqrt{1-v^2}\right) = \ln cx$$

$$\ln\left(1+\sqrt{1-v^2}\right) = -\ln cx$$

$$\ln\left(1+\sqrt{1-v^2}\right) = \ln(cx)^{-1}$$

$$1+\sqrt{1-v^2} = \frac{1}{cx}$$

$$1+\sqrt{\frac{x^2-y^2}{x^2}} = \frac{1}{cx}$$

$$x+\sqrt{x^2-y^2} = \frac{1}{c}$$

$$\boxed{x+\sqrt{x^2-y^2} = c_1} \qquad \because \frac{1}{c} = c_1$$

Which is required solution.

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Q3. $(D^4+D^2)y = 3x^2 + 4\sin x - 2\cos x$

Solution:-

$$(D^4+D^2)y = 3x^2 + 4\sin x - 2\cos x$$

$$\Rightarrow \quad f(D)y = f(x)$$

As it is non-homogeneous linear equation
So solution will be

$$y = y_c + y_p \quad - (i)$$

Complementary solution y_c

$$D^4 + D^2 = 0 \quad \Rightarrow D^2(D^2+1) = 0$$

Either $D^2 = 0 \Rightarrow \boxed{D=0}$

$$D^2 + 1 = 0 \quad \Rightarrow D^2 = -1$$

$$D = \sqrt{-1} \quad \Rightarrow D = i \quad \text{or} \quad D = 0 + i$$

Roots are real and complex

$y_c = c_1 e^{0x} + e^{0x}(c_2 \cos x + c_3 \sin x)$

$y_c = c_1 + c_2 \cos x + c_3 \sin x$

$y_p = \frac{1}{f(D)} F(x)$

$y_p = \frac{1}{D^4 + D^2}(3x^2 + 4\sin x - 2\cos x)$

$= \frac{3x^2}{D^4+D^2} + \frac{4\sin x}{D^4+D^2} - \frac{2\cos x}{D^4+D^2}$

$f(D) = D^4 + D^2$

at $D=0 \Rightarrow f(D) = 0$

So $f'(D) = 4D^3 + 2D$

~~So~~ Now also for $D=0 \Rightarrow f'(D) = 0$

again differentiating

$f''(D) = 12D^2 + 2$

So for $D=0$

$f''(0) = 12(0)+2 = 2$

So replacing $\frac{1}{f(D)}$ with $\frac{x^2}{f''(D)}$

$\Rightarrow y_p = \frac{x^2 \, 3x^2}{12D+2} + \frac{x^2}{12D+2} \cdot 4\sin x - \frac{x^2}{12D+2} \cdot 2\cos x$

Putting $D=0$ in all

So

$y_p = \frac{x^2 3x^2}{12(0)+2} + \frac{x^2 4\sin x}{12(0)+2} - \frac{2x^2\cos x}{12(0)+2}$

$y_p = \frac{3x^4}{2} + \frac{4x^2\sin x}{2} - \frac{2x^2\cos x}{2}$

$= \frac{3}{2}x^4 + 2x^2\sin x - x^2\cos x$

So

Putting in eq $\rightarrow$ 1

$y = C_1 + C_2\cos x + C_3\cos x + \frac{3}{2}x^4 + 2x^2\sin x$

$- x^2\cos x$

$$\boxed{y = C_1 + (C_2 - x^2)\cos x + (C_3 + 2x^2)\sin x + \frac{3}{2}x^4}$$

Ans