DIFFERENTIAL EQUATIONS

Submitted by

Mohsin Ali Khan

ID 7426

Section B

Submitted to Shoumaila Mazhar

IQRA NATIONAL UNIVERSITY PESHAWAR

Q(2)

(3)

(2)

$$\frac{dy}{dt} = e^{y-t} \sec(y)(1+t^2)$$

$y(0)=0$ so $x=0$ $y=0$

$$dy = e^{y} \cdot e^{-t} \sec(y)(1+t^2)\,dt$$

$$\frac{1}{e^{y} \cdot \sec(y)}\,dy = (1+t^2)e^{-t}\,dt$$

As $\cos(y) = 1/\sec(y)$

$$\int e^{-y} \cos y\,dy = \int (1+t^2)e^{-t}\,dt$$

using integration by parts

$$e^{-y}\int \cos y\,dx - \int\left(\int \cos y \cdot \frac{d}{dy}e^{-y}\right) = (1+t^2)\int e^{-t} - \int$$

$$\left(\int e^{-t} \cdot \frac{d}{dt}(1+t^2)\right) \rightarrow eq\ 1$$

L.H.S

$$e^{-y}\int \cos y\,dx - \int\left(\int \cos y \cdot \frac{d}{dy}e^{-y}\right)$$

$e^{-y}\sin y - \int(\sin y \cdot e^{-y}(-1))$

$e^{-y}\sin y + \int(\sin y \cdot e^{-y})$

$e^{-y}\sin y + \int(e^{-y}\sin y)$

Again using integration by parts

$e^{-y}\sin y + e^{-y}(-\cos y) - \int(\int \sin y \frac{d}{dy} e^{-y})$

$e^{-y}\sin y + e^{-y}(-\cos y) - \int(-\cos y \frac{e^{-y}}{-1})$

$e^{-y}\sin y - e^{-y}\cos y - \int(\cos y e^{-y})$

$\int(\cos y e^{-y}) = LHS$

Since it is given same to the first one so L.H.S will become

$LHS = e^{-y}(\sin y - \cos y) - L.H.S$

$2LHS = e^{-y}(\sin y - \cos y)$

$LHS = \underline{e^{-y}(\sin y - \cos y)}$

Now taking R.H.S

$\int (1+t^2) e^{-t} dt$

$= (1+t^2) \int e^{-t} - \int \left(\int e^{-t} \cdot \frac{d}{dt} (1+t^2) \right)$

$= (1+t^2) e^{-t} - \int \left(-e^{-t} (2t) \right)$

$= -(1+t^2) e^{-t} + \int \left((2t) e^{-t} \right)$

Again using integration by parts

$= -(1+t^2) e^{-t} + \left(2t \int e^{-t} - \int \left(e^{-t} \frac{d}{dt} 2t \right) \right)$

$= -(1+t^2) e^{-t} + \left(-2t e^{-t} - \int \left(-e^{-t} 2 \right) \right)$

$= -(1+t^2) e^{-t} + \left(-2t e^{-t} + \int \left(2e^{-t} \right) \right)$

$= -(1+t^2) e^{-t} + \left(-2t e^{-t} - 2e^{-t} \right) + c$

$= -(1+t^2) e^{-t} - 2t e^{-t} - 2e^{-t} + c$

$= -e^{-t} - e^{-t} t^2 - 2t e^{-t} - 2e^{-t} + c$

$= -(t^2 + 2t + 3) e^{-t} + c = R.H.S$

Now look L.H.S = R.H.S

$$\frac{e^{-y}(\sin y-\cos y)}{2} = -(t^2+2t+3)e^{-t}+c$$

we know that

$t=0$ $y=0$

put it above

$$\frac{1}{2}(0-1) = -3+c$$

$$c = 5/2$$

put value of c

$$\boxed{\frac{e^{-y}}{2}(\sin y-\cos y) = -(x^2+2t+3)e^{-t}+5/2}$$

Q2) $(\sqrt{x+y} + \sqrt{x-y}\, dx - (\sqrt{x+y} - \sqrt{x-y})\, dy = 0$ ⑤

$$\frac{dy}{dx} = \frac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x+y} - \sqrt{x-y}} \longrightarrow ①$$

This is Homogeneous Differential eq in x and y to solve this put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

This eq ① becomes

$$v + x\frac{dv}{dx} = \frac{\sqrt{x+vx} + \sqrt{x-vx}}{\sqrt{x+vx} - \sqrt{x-vx}}$$

$$v + x\frac{dv}{dx} = \frac{\sqrt{1+v} - \sqrt{1-v}}{\sqrt{1+v} - \sqrt{1-v}}$$

$$v + x\frac{dv}{dx} = \frac{\sqrt{1+v} + \sqrt{1-v}}{\sqrt{1+v} - \sqrt{1-v}} \times \frac{\sqrt{1+v} + \sqrt{1-v}}{\sqrt{1+v} + \sqrt{1-v}}$$

$$v + x\frac{dv}{dx} = \frac{1 + \cancel{v} + 1 - \cancel{v} + 2\sqrt{1-v^2}}{2v}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{2}(1 + \sqrt{1-v^2})}{\cancel{2}v}$$

$$v + x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}}{v}$$

$$x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}}{v} - v$$

$$x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2} - v^2}{v}$$

$$x\frac{dv}{dx} = \frac{\sqrt{1-v^2}\,(1+\sqrt{1-v^2})}{v}$$

$$x\frac{dv}{dx} = \frac{\sqrt{1-v^2}\,(1+\sqrt{1-v^2})}{v}$$

$$\frac{v\,dv}{\sqrt{1-v^2}\,(1+\sqrt{1-v^2})} = \frac{dx}{x}$$

taking integral on b/s

$$\int \frac{x\,dv}{\sqrt{1-v^2}\,(1+\sqrt{1-v^2})} = \int \frac{dx}{x}$$

put $1+\sqrt{1-v^2} = t$

$$\Rightarrow \tfrac{1}{2}(1-v^2)^{-1/2}(-2v)\,dv = dt$$

$$\frac{v\,dv}{\sqrt{1-v^2}} = -dt$$

$$\int -\frac{dt}{t} = \int \frac{dx}{x}$$

$$-\ln t = \ln x + \ln c$$

$$-\ln(1+\sqrt{1-v^2}) = \ln cx$$

$$\ln(1+\sqrt{1-v^2}) = -\ln cx$$

$$\cancel{\ln}(1+\sqrt{1-v^2}) = \cancel{\ln}(cx)^{-1}$$

$$1+\sqrt{1-v^2} = 1/cx$$

$$1+\sqrt{1-\frac{y^2}{x}} = \frac{1}{cx}$$

$$1+\sqrt{\frac{x^2-y^2}{x^2}} = 1/cx$$

$$x+\sqrt{x^2-y^2} = 1/c$$

$$x+\sqrt{x^2-y^2} = c_1 \quad \because \frac{1}{c} = c_1$$

which is a Required solution.

Q.No(3) $(D^4 + D^2)y = 3x^2 + 4\sin x - 2\cos x$ (8)

Solution:~

$$(D^4 + D^2)y = 3x^2 + 4\sin x - 2\cos x$$

$$\Rightarrow f(D)y = f(x)$$

As it is non-homogenous linear equation so solution will be

$$y = y_c + y_p \quad -(i)$$

complementary solution y_c

$$D^4 + D^2 = 0 \Rightarrow D^2(D^2+1) = 0$$

Either $D^2 = 0 \Rightarrow D = 0$

$$D^2 + 1 = 0 \Rightarrow D^2 = -1$$

$$D = \sqrt{-1} \Rightarrow \boxed{D = i}, \text{ or } D = \boxed{0+i}$$

Roots are real and complex

$$y_c = c_1 e^{0x} + e^{0x}(c_2 \cos x + c_3 \sin x)$$

$$y_c = c_1 + c_2 \cos x + c_3 \sin x$$

$$y_p = \frac{1}{f(D)} F(x)$$

$$y_p = \frac{1}{D^4 + D^2} (3x^2 + 4\sin x - 2\cos x)$$

$$= \frac{3x^2}{D^4 + D^2} + \frac{4\sin x}{D^4 + D^2} - \frac{2\cos x}{D^4 + D^2}$$

$$f(D) = D^4 + D^2$$

at $D = 0 \Rightarrow f(D) = 0$

So $f'(D) = 4D^3 + 2D$

Now also for $D = 0 \Rightarrow f'(D) = 0$

again differentiating

$$f''(D) = 12D + 2$$

So for $D = 0$

$$f''(0) = 12(0) + 2 = 2$$

So replacing $\frac{1}{f(D)}$ with $\frac{x^2}{f''(D)}$

$$\Rightarrow y_p = \frac{x^2 \, 3x^2}{12D + 2} + \frac{x^2}{12D+2} \cdot 4\sin x - \frac{x^2}{12D+2} \cdot 2\cos x$$

Putting $D=0$ in all

so

$$y_p = \frac{x^2\, 3x^2}{12(0)+2} + \frac{x^2\, 4\sin x}{12(0)+2} - \frac{2x^2\cos x}{12(0)+2}$$

$$y_p = \frac{3x^4}{2} + \frac{4x^2\sin x}{2} - \frac{2x^2\cos x}{2}$$

$$= \frac{3}{2}x^4 + 2x^2 \quad x - x^2\cos x$$

so putting in equation (i)

$$y = c_1 + c_2\cos x + c_3\cos x + 3/2\, x^4 + 2x^2\sin x - x\cos$$

$$\boxed{y = c_1 + (c_2 - x^2)\cos x + (c_3 + 2x^2)\sin x + 3/2\, x^4}$$