Name: Aftab Khan **ID:12985**
Subject: Differential Equation

1

Q1 (a):

Definition:

A second order differential involving y, y' and y'' with these multiple only function of x is called linear.

i.e. $a(x)y'' + b(x)y' + c(x)y = d(x)$

is called second order linear differential equation.

* **Homogenous:** when $d(x) = 0$

i.e $a(x)y'' + b(x)y' + c(x)y = 0$

Example: $y'' + xy' + 2y = 0$

* **non-homogenous:** when $d(x) \neq 0$

$a(x)y'' + b(x)y' + c(x)y = dx$

Example: $y'' + xy' + 2y = x^2 + 3$

Q1 a: (i) $16y'' + 24y' + 9y = 0$

Sol: Given D.E $16y'' + 24y' + 9y = 0$

Characteristic of given D.E $16m^2 + 24m + 9 = 0$

Evaluate the roots of equation as follows

$$16m^2 + 24m + 9 = 0$$
$$(4m+3)^2 = 0$$
$$4m = -3$$
$$m = -\frac{3}{4}$$

Therefore root equation are $m_1 = m_2 = m = -\frac{3}{4}$

Since roots of characteristic equation are real and equal, solution must be in the form of $y = C_1 e^{mx} + C_2 x e^{mx}$

Therefore, solution is $\boxed{y = C_1 e^{-\frac{3}{4}x} + C_2 x e^{-\frac{3}{4}x}}$

(ii) Q: $y'' - 4y' - 12y = 3e^{5x}$

Sol: Given D.E. $y'' - 4y' - 12y = 3e^{5x}$

Consider homogenous part of equation $y'' - 4y' - 12y = 0$

Characteristic of D.E is $m^2 - 4m - 12 = 0$

Evaluate the roots of equation as follows

$m^2 - 4m - 12 = 0$

$m^2 - 6m + 2m = 0$

$(m-6)(m+2) = 0$

$m_1 = 6,\ m_2 = -2$

Therefore, root of equation are $m_1 = 6$, $m_2 = -2$

Root of equation are real & distinct, solution must be in form of $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

Therefore solution is $\boxed{y_h = C_1 e^{6x} + C_2 e^{-2x}}$

To find Particular solution, let $y = pe^{5x}$ be the solution

Substitute the solution $y = pe^{5x}$ in equation as follow

$$(pe^{5x})'' - 4(pe^{5x})' - 12(pe^{5x}) = 3e^{5x}$$

$$25pe^{5x} - 20pe^{5x} - 12pe^{5x} = 3e^{5x}$$

$$-7pe^{5x} = 3e^{5x}$$

$$p = -\frac{3}{7}$$

Thus, chosen solution becomes $y = -\frac{3}{7}e^{5x}$

Evaluate general solution as follow

$$y = y_h + y_p$$

$$y = C_1 e^{6x} + C_2 e^{-2x} - \frac{3}{7}e^{5x}$$

Therefore, solution of DE is $\boxed{y = C_1 e^{6x} + C_2 e^{-2x} - 37e^{5x}}$

Q2 :- (i) $2y'' + 5y + 3y = 0$ $y(0) = 3$ $y'(0) = -4$

Sol:- Given that

$2y'' + 5y' + 3y = 0$ with initial conditions

$\Rightarrow y(0) = 3,\ y'(0) = 4$

The auxiliary equation of given equation is

$$2m^2 + 5m + 3 = 0$$

Solving equation by formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore,

$$m = \frac{-5 \pm \sqrt{25 - 4 \times 6}}{4}$$

$$= \frac{-5 \pm \sqrt{1}}{4}$$

$$= \frac{-5 \pm 1}{4}$$

So $m_1 = -1$ or $m_2 = \frac{-3}{2}$

~~The~~ Therefore solution given by

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

That is $y = C_1 e^{-x} + C_2 e^{-\frac{3}{2}x}$

Applying initial condition $y(0) = 3$, $y'(0) = -4$

$y(0) = 3 \Rightarrow C_1 + C_2 = 3$ .... (1)

$y'(x) = -C_1 e^{-x} - \frac{3}{2} C_2 e^{\frac{-3}{2}x}$

$y'(0) = -4 \Rightarrow -C_1 - \frac{3}{2} C_2 = 4$ ...... (2)

(1) + (2)

$\Rightarrow C_2 - \frac{3}{2} C_2 = -1$

$\Rightarrow C_2 = 2$

Hence $C_1 = 1$

Therefore $\boxed{y = e^{-x} + 2e^{\frac{-3}{2}x}}$ is solution

(ii) $2y'' + 5y' - 3y = 0 \quad y(0) = 3 \quad y'(0) = 4$

Sol:- $2y'' + 5y' - 3y = 0$, intial condition $y(0) = 3$ & $y'(0) = 4$

auxiliary equation of give D.E is

$2m^2 + 5m - 3 = 0$

Solve equation by factorization

$2m^2 + 5m - 3 = 0$

$2m^2 + 6m - m - 3 = 0$

$2m(m+3) - 1(m+3) = 0$

$(m+3)(2m-1) = 0$

Therefore,

$m_1 = -3$ & $m_2 = \frac{1}{2}$

Hence solution of D.E is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$y = C_1 e^{-3x} + C_2 e^{\frac{1}{2}x}$$

Applying initial condition

$y(0) = 3$

$\Rightarrow C_1 + C_2 = 3 \quad \ldots\ldots (1)$

$y' = -3C_1 e^{-3x} + \frac{1}{2} C_2 e^{\frac{1}{2}x}$

$y'(0) = 4$

$\Rightarrow -3C_1 + \frac{1}{2} C_2 = 4 \quad \ldots\ldots (2)$

$3 \times (1) + (2)$ implies

$\left(3 + \frac{1}{2}\right) C_2 = 13$

$C_2 = \frac{26}{7}$

Therefore, $C_1 = \frac{-5}{7}$

Therefore, solution of IVP is $\boxed{y = \frac{26}{7} e^{-3x} - \frac{5}{7} e^{\frac{1}{2}x}}$

(iii) $y'' - 4y' + 9y = 0$. $y(0) = 0$, $y'(0) = -8$

Soln

Given

$$y'' - 4y' + 9y = 0 \qquad y(0) = 0 \; , \; y'(0) = -8$$

Now let take $m = \frac{d}{dx}$ then equation becomes

$$m^2 - 4m + 9 = 0$$

if equation is form ax^2+bx+c then root equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so roots for $m^2 - 4m + 9 = 0$ is

$$m = \frac{4 \pm \sqrt{16 - 36}}{2}$$

$$= \frac{4 \pm \sqrt{-20}}{2}$$

$$= 2 \pm i\sqrt{5}$$

Hence, solution for equation is

$$y = e^{2t}\left(A\cos\sqrt{5}t + B\sin\sqrt{5}t\right)$$

we know that

$$y(0) = 0$$

$$0 = e^{0}(A)$$

$$A = 0$$

Hence $y = e^{2t}\left(B\sin\sqrt{5}t\right)$

Given $y'(0) = -8$

$$y' = 2\sqrt{5}\,e^{2t}\,B\cos\sqrt{5}t$$

$$y'(0) = 2\sqrt{5}B$$

$$-8 = 2\sqrt{5}B$$

$$B = \frac{-4}{\sqrt{5}}$$

The solution is $\boxed{y = \frac{-4}{\sqrt{5}}e^{2t}\cos\sqrt{5}t}$

Q3: Define laplace transform:

laplace transform is the integral of the given derivative function with real "t" to convert into complex function with variable S. for $t \geq 0$, let $f(t)$ be given and assume the function satisfies certain condition to be stated later on.
The laplace transform of $f(t)$, it is denoted by $f(t)$ or $f(s)$.

Examples:

1. let $f(t) = 1$, $t \geq 0$

2. let $f(t) = \sin at$, $t \geq 0$

Find laplace transform of function

Q3 A ① :

$$f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$$

Sol:

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

②

$$g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$$

Sol:

$$G(s) = 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

③ $h(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t)$

Sol:

$$H(s) = \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

Q4 :- (a) Solve IVP using laplace transform

(i) :- $y'' - 4y' = e^{3t}$ $y(0) = 0,\ y'(0) = 0$

Sol :- Take

$$y(t) \rightarrow Y(s)$$

$$y'(t) \rightarrow sY(s) - y(0)$$

$$y''(t) \rightarrow s^2Y(s) - sy(0) - y'(0)$$

Now taking laplace $y'' - 4y' = e^{3t}$

$$s^2Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) = \frac{1}{s-3}$$

$$s^2Y(s) - s\times 0 - 0 - 4sY(s) + 4\times 0 = \frac{1}{s-3}$$

$$s^2Y(s) - 4sY(s) = \frac{1}{s-3}$$

$$Y(s)(s^2 - 4s) = \frac{1}{s-3}$$

$$Y(s) = \frac{1}{s(s-4)(s-3)}$$

Hence $\frac{1}{s(s-4)(s-3)} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s-3}$

$1 = A(s-4)(s-3) + B(s)(s-3) + C(s)(s-4)$

Take $s=0$

$1 = A(-4)(-3)$

$A = \frac{1}{12}$

Take $s=4$

$1 = B \times 4 \times 1$

$B = \frac{1}{4}$

Take $S=3$

$1 = C \times 3 \times -1$

$C = \frac{-1}{3}$

Hence

$Y(s) = \frac{1}{12}\frac{1}{s} + \frac{1}{4}\frac{1}{s-4} - \frac{1}{3}\frac{1}{s-3}$

solution for IVP is

$\boxed{y(t) = \frac{1}{12} + \frac{1}{4}e^{4t} - \frac{1}{3}e^{3t}}$

Q4: (ii) $y'' + 3y' + 2y = e^{-t}$ $y(0) = 0$ $y'(0) = 0$

Sol: Given equation & initial values

$$y'' + 3y' + 2y = e^{-t} \ \& \ y(0) = 0, \ y'(0) = 0$$

Take laplace transformation

$$L[y''] + 3L[y'] + 2L[y] = L[e^{-t}]$$

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1}$$

Put initial value & simply

$$s^2Y(s) + 3[sY(s)] + 2Y(s) = \frac{1}{s+1}$$

$$Y(s)[s^2 + 3s + 2] = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{s+1[s^2 + 3s + 2]}$$

$$= \frac{1}{(s+1)^2 (s+2)}$$

Now find partial fraction of tranformation

$$Y(s) = -\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}$$

Since, we have to teem of S+1. So take one with t multiple and find transformation

$$\boxed{y(t) = -e^{-t} + te^{-t} + e^{-2t}}$$ Answer