

# FINAL TERM EXAM

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Question 1) Find the Fourier series representation of  $f(t) = 1+t$   
 $-\pi \leq t \leq \pi$

Solution:

(1)

Here we use the formula  
 $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$  - eq (1)

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+t) dt$$

$$a_0 = \frac{1}{2\pi} \left[ t + \frac{t^2}{2} \right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi} \left( \pi - (-\pi) + \frac{\pi^2}{2} - \left( -\frac{\pi^2}{2} \right) \right)$$

$$a_0 = \frac{1}{2\pi} \left( 2\pi + \frac{2\pi^2}{2} \right)$$

$$a_0 = \frac{1}{2\pi} (2\pi + \pi^2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) (\cos nt) dt$$

$$a_n = \frac{1}{\pi} \left( (1+t) \frac{\sin nt}{n} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left( \frac{\sin nt}{n} \frac{d}{dt} (1+t) \right) \right)$$

$$a_n = \frac{1}{\pi} \left( (1+t) \frac{\sin nt}{n} \Big|_{-\pi}^{\pi} - \frac{\cos nt}{n^2} \Big|_{-\pi}^{\pi} \right)$$

$$a_n = \frac{-1}{n^2 \pi} (\cos n\pi - \cos n(-\pi))$$

$$a_n = \frac{-1}{n^2 \pi} (-1 - (-1))$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) \sin nt \, dt \quad (2)$$

$$b_n = \frac{1}{\pi} \left( (1+t) \int_{-\pi}^{\pi} \sin nt - \int_{-\pi}^{\pi} \left( \int \sin nt - \frac{d}{dt} (1+t) dt \right) \right)$$

$$b_n = \frac{1}{\pi} \left( \frac{(1+t)(-\cos nt)}{n} \right) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left( -\frac{\cos nt}{n} \right) dt$$

$$b_n = \frac{1}{\pi} \left( -\frac{(1+t)(\cos nt)}{n} \right) \Big|_{-\pi}^{\pi} + \left( \frac{\sin nt}{n^2} \right) \Big|_{-\pi}^{\pi}$$

$$b_n = \frac{-1}{n\pi} \left( (1+\pi)(\cos n\pi) - (1+(-\pi))(\cos n(-\pi)) \right)$$

$$b_n = \frac{-1}{n\pi} \left( \cancel{\cos n\pi} + \pi \cos n\pi - \cancel{\cos n\pi} + \pi \cos n\pi \right)$$

$$b_n = \frac{-1}{n\pi} (2\pi \cos n\pi)$$

Here  $\cos n\pi = \frac{(-1)^{n+1}}{n}$

$$b_n = \frac{2}{n} (-1)^{n+1}$$

So equation become

$$f(x) = \frac{1}{2\pi} (2\pi + \pi^2) + 0 + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nt$$

Question 2) Calculate the characteristic equation the Eigen values of the system. Where A is given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix}$$

(3)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix}$$

Eigen Values = ?

Solution:

Step = 1

We have;

$$(A - \lambda I)X = 0 \quad A = \text{Given Matrix}$$

Step = 2

We have; The characteristic equation is given by,

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 1-\lambda & 4 \\ 0 & 2 & 2-\lambda \end{vmatrix} = 0$$

Step = 3

$$\lambda^3 - \left| \begin{matrix} \text{Sum of} \\ \text{Diagonal element} \end{matrix} \right| \lambda^2 + \left| \begin{matrix} \text{Sum of} \\ \text{Diagonal} \\ \text{minors} \end{matrix} \right| \lambda - |A| = 0 \quad \text{--- (B)}$$

$$\text{Sum of Diagonal elements} = 1 + 1 + 2 = 4$$

$$\begin{aligned}\text{Sum of Diagonal minors} &= \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \\ &= (-6) + (2) + (1) \\ &= -6 + 2 + 1 \quad \text{--- } \textcircled{4}\end{aligned}$$

By putting values in eq (B)

$$\lambda^3 - 4\lambda^2 - 3\lambda - |A| = 0 \quad \text{--- } \textcircled{C}$$

$$\begin{aligned}|A| &= \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \\ &= 1(2-8) - 0 + 1(6-0) \\ &= -6 + 6 \\ &= 0\end{aligned}$$

By putting values in (o)

$$\lambda^3 - 4\lambda^2 - 3\lambda - 0 = 0$$

$$\lambda^3 - 4\lambda^2 - 3\lambda = 0$$

$$\lambda^2(\lambda^2 - 4\lambda - 3) = 0$$

$$\lambda = 0$$

$$\lambda^2 - 4\lambda - 3 = 0$$

(5)

using Quadratic Formula;

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 + 12}}{2} = \frac{4 \pm \sqrt{28}}{2}$$

$$\lambda = \frac{4 + \sqrt{28}}{2}, \lambda = \frac{4 - \sqrt{28}}{2}$$

$$\begin{aligned} a &= 1 \\ b &= -4 \\ c &= -3 \end{aligned}$$

we have eigenvalues;

$$\lambda = \left( 0, \frac{4 + \sqrt{28}}{2}, \frac{4 - \sqrt{28}}{2} \right) \text{ required solution}$$

Question 3) Solve the following system of linear equations

$$5x + 0 + 4z + 2m = 3$$

$$x - y + 2z + m = 1$$

$$4x + y + 2z + 0 = 1$$

$$x + y + z + m = 0$$

(6)

Solution:

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right] \quad \underbrace{R_4 R_2}$$

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 0 & 2 & -1 & 0 & -1 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -1 & +4/5 & +4/5 & 3/5 \\ 0 & 2 & -1 & 0 & -1 \end{array} \right] \quad \underbrace{-1/5 \times R_3}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right] \quad \underbrace{C_2 \times -5}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 0 & 6 & 1 & 2 \\ 0 & 0 & 0 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & -5 & 24/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

(7)

$$\left[ \begin{array}{cccc|c} 1 & 0 & 1 & 1/2 & 3/4 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/2 \end{array} \right] \quad \underbrace{5/4 \times R_1}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 126/84 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -1 & 6/5 & 4/5 & 3/5 \\ 0 & 0 & 7/5 & 8/5 & 1/5 \end{array} \right] \quad \underbrace{5 \times R_3, \text{ and } 5 \times R_4}$$

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \underbrace{5R_3, \text{ and } 5R_4}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \underbrace{1/5 \times R_1}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -1 & 6/5 & 1/5 & 2/5 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \underbrace{R_2 \times 5}$$



$$\left[ \begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad R_3 - R_2$$

(8)

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 2 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 8/7 & 1/7 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right] \quad \begin{array}{l} R_3 - R_4 \\ 1/7 \times R_3 \\ 1/3 \times R_4 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 3/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$(x, y, z, m) = (3/4, 31/21, -11/21, 1/3)$$

$$x = 3/4$$

$$y = 31/21$$

$$z = -11/21$$

$$m = 1/3$$

Question 4) verify that

$$u(x,t) = \sin(x+2t)$$

(9)

is a solution of the one-dimensional wave equation.

Solution: Given Data

$$u(x,t) = \sin(x+2t)$$

Differentiate w.r.t  $x$  partially

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \sin(x+2t)$$

$$\frac{\partial u}{\partial x} = \cos(x+2t) \cdot \frac{\partial}{\partial x} (x+2t)$$

$$\frac{\partial u}{\partial x} = \cos(x+2t) (1+0)$$

$$\frac{\partial u}{\partial x} = \cos(x+2t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \cos(x+2t)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t) \cdot \frac{\partial}{\partial x} (x+2t)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t) (1+0)$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t)}$$

(1)

and

$$u(x,t) = \sin(x+2t)$$

Differentiate w.r.t "t".

(10)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \sin(x+2t)$$

$$\frac{\partial u}{\partial t} = \cos(x+2t) (0+2)$$

$$\frac{\partial u}{\partial t} = 2 \cos(x+2t)$$

$$\frac{\partial^2 u}{\partial t^2} = (2) \cdot \sin(x+2t) (0+2)$$

$$\frac{\partial^2 u}{\partial t^2} = -4 \sin(x+2t)$$

As we know that one-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$-4 \sin(x+2t) = c^2 [-\sin(x+2t)]$$

$$-4 \sin(x+2t) = -c^2 \sin(x+2t)$$

$$-4 \sin(x+2t) + c^2 \sin(x+2t) = 0$$

For the arbitrary constant  $c = \pm 2$

$$-4 \sin(x+2t) + (\pm 2)^2 \sin(x+2t) = 0$$

$$-4 \sin(x+2t) + 4 \sin(x+2t) = 0$$

$$0 = 0$$

~~xxxx~~ Then it will be verified for the arbitrary constant  $c = 2$