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Sec A-

Differential Equations

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QNo (1):

$$\frac{\partial w}{\partial t^2} = c\frac{\partial^2 w}{\partial x^2}$$

$w(\sin x+ct)+\cos(2x+2ct)$.

(i) Solution:

$w = \sin(x+ct)+\cos(2x+2ct)$

Partial derivative.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial t}\left(\sin(x+ct)+\cos(2x+2ct)\right).$$

$$\frac{\partial w}{\partial t} = \cos(x+ct)+c-\sin(2x+2ct)+2c.$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct)+c^2-\cos(2x+2ct)+4c^2 \rightarrow (a)$$

$$\frac{\partial w}{\partial t} = \cos(x+ct)-\sin(2x+2ct)+2.$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct)-4\cos(2x+2ct).$$

$$= +c^2\left[-\sin(x+ct)-4\cos(x+2ct)\right].$$

$$\frac{\partial^2 w}{\partial t^2} = +c^2\left[-\sin(x+ct)-4\cos(2x+2ct)\right]$$

$$\boxed{c^2 * \frac{\partial^2 w}{\partial x^2}} \text{ or } \underline{c^2\cdot\frac{\partial^2 w}{\partial x^2}}$$

$w = \tan(2x + ct)$.

(ii) Solution:-

$w = \tan(2x + ct)$.

Differentiating w.r.t "t"

$$\frac{\partial w}{\partial t} = c \sec^2(2x+ct).$$

again Differentiating

$$\frac{\partial^2 w}{\partial t} = \frac{\partial}{\partial t}\left(c \sec^2(2x+ct)\right)$$

$$= 2c^2 \cdot 2\sec^2(2x+ct) \tan 2x+ct.$$

Now by differentiating w.r.t x

$$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

again Differentiating.

$$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct).$$

Now

$= 4c^2 \sec^2(2x+ct)\tan(2x+ct)$.

$= 4c^2 \sec^2(2x+ct)\tan(2x+ct)$

$0 = 0$

Hence $w = \tan(2x+ct)$ is the solution of given equation.

Q(b)

$$F(x) = x \; ; \quad -\pi < x \leq 0$$
$$2x \; ; \quad 0 \leq x \leq \pi.$$

Solution:

$$F(x) = \begin{cases} x \; ; & -\pi < x \leq 0 \\ 2x \; ; & 0 \leq x \leq \pi \end{cases}$$

Now we will find a_0, a_n and b_n.

~~$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$~~

Now

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} x\,dx + \frac{1}{\pi} \int_{0}^{\pi} 2x\,dx$$

$$a_0 = \frac{1}{\pi} \left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \frac{1}{\pi} \left[0 - \frac{\pi^2}{2}\right] + \frac{2}{\pi} \left[\frac{\pi^2}{2} - 0\right]$$

$$\boxed{a_0 = \frac{-\pi}{2} + \pi = \pi/2} \text{ — (b)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(x\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi}(2x\cos nx)\,dx$$

$$a_n = \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos nx}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos nx}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}.$$

Hence

$$a_n = \begin{cases} \frac{-2}{\pi n^2} & ; \text{ if } n \text{ is odd number} \\ 0 & ; \text{ if } n \text{ is even number.} \end{cases}$$

Now

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}.$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right]$$

$$= -3\frac{\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}$$

Hence the required fourier Series is given as

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$$

$$= \pi/4 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n}.$$

Answer

Qno 8) Solve initial Value Problem Page (7)

$$y'' + 4y' + 13y = 8\sin 3x \;;\; y(0)=1,\; y'(0)=2.$$

Solution:-

$$y'' - 4y' + 13y = 8\sin 3x \quad \text{— (a)} \quad y(0)=1,\; y'(0)=2$$

So Associated homogenous equation of (a) is given by

$$y'' - 4y' + 13y = 0 \quad \text{— (b)}$$

changing eq (b) into A.E.

Putting $y = e^{mx}$ in eq (b).

$$m^2 - 4m + 13 = 0.$$

By Quadratic formula.

$$a=1 \;;\; b=-4,\; c=13.$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2-4(1)(13)}}{2(1)}$$

$$m = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}.$$

$\cancel{m = 2 \pm 3i}$ $m = 2 \pm 3i$

$$m_1 = 2+3i, \quad m_2 = 2-3i.$$

$$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) \text{ --- (c)}$$

let $y_p = A \cos 3x + B \sin 3x$ --- (d)

Differentiating w.r.t "x"

$$y'_p = -3A \sin 3x + 3B \cos 3x.$$

Again Differentiating w.r.t "x"

$$y''_p = -9A \cos 3x - 9B \sin 3x.$$

Putting the Values in eq (a)

$$\text{eq (a)} \Rightarrow (-9A \cos 3x - 9B \sin 3x) - 4(-3A \sin 3x + 3B \cos 3x)$$

$$+ 13(A \cos 3x + B \sin 3x) = 8 \sin 3x.$$

$$\Rightarrow -9A \cos 3x - 12B \cos 3x + 13A \cos 3x - 9B \sin 3x + 12A \sin 3x + 13B \sin 3x$$

$$= 8 \sin 3x.$$

$$\Rightarrow (-9A - 12B + 13A) \cos 3x + (-9B + 12A + 13B) \sin 3x = 8 \sin 3x$$

Comparing Co-efficients

$\sin 3x \Rightarrow 4B + 12A = 8$ --- (e).

$\cos 3x \Rightarrow 4A - 12B = 0 \Rightarrow 4A = 12B \rightarrow$ (f)

$$\text{eq (f)} \Rightarrow \boxed{A = 3B} \text{ --- (g)}$$

Putting the Value of A from eq (g) in eq (e).

$$4B + 12(3B) = 8.$$

$$4B + 36B = 8.$$

$$40B = 8.$$

$$\boxed{B = 1/5} \text{ --- (h)}$$

Putting the Value of B in eq (g).

$$A = 3(1/5) \Rightarrow \boxed{A = 3/5}$$

Putting the Value of A and b in eq (d)

$$y_p = (3/5 \cos 3x) + 1/5 \sin(3x)$$

The General Solution is given by.

$$y = y_c + y_p.$$

$$y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) + 3/5 \cos 3x.$$
$$+ 1/5 \sin 3x \text{ --- (i)}$$

Now for the the Values of C_1 and C_2.

Putting $x = 0$ and $y = 1$ in eq (i)

$$1 = e^{2(0)}\left(\left(C_1 \cos 3(0) + C_2 (\sin 3(0)\right)\right) + 3/5 \cos 3(0) + 1/5 \sin 3(0).$$

$$1 = \left(\left(C_1 (1) + C_2(0) + 3/5(1) + 1/5(0)\right)\right)$$

$$1 = C_1 + 3/5$$

$$C_1 = 1 - 3/5 \Rightarrow 2/5 \text{ —— } (j)$$

Differentiating e w.r.t "x"

$$y' = C_1\left(2e^{2x}\cos 3x - 3e^{2x}\sin 3x\right) + C_2\left(2e^{2x}\sin 3x + 3e^{2x}\cos 3x\right)$$
$$- 6/5 \sin 3x + 3/5 \cos 3x \text{ —— } (K)$$

Putting $y' = 2$; $x = 0$ in eq (K)

$$y' = C_1\left(2e^{2x}\cos 3x - 3e^{2x}\sin 3x\right) + C_2\left(2e^{2x}\sin 3x + 3e^{2x}\cos 3x\right). - 6/5 \sin 3x + 3/5 \cos 3x$$

Putting $y' = 2$; $x = 0$.

$$2 = C_1\left(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0)\right) + C_2\left(2e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0)\right) - 6/5 \sin 3(0) + 3/5 \cos 3(0).$$

$$2 = C_1(2) + C_2(3) - 0 + 3/5$$

$$2 = 2C_1 + 3C_2 + 3/5$$

Putting $C_1 = 2/5$.

$$2 = 2(2/5) + 3C_2 + 3/5.$$

$$2 = 7/5 + 3C_2.$$

$$3C_2 = 2 - 7/5 \Rightarrow 3C_2 = 3/5$$

$$\boxed{C_2 = 3/15} \text{ --- eq (L)}$$

Putting the above values we get.

$$y = e^{2x}\left(2/5 \cos 3x + 3/15 \sin 3x\right) + 3/5 \cos 3x + 1/5 \sin 3x$$

$$\boxed{y = 2/5\, e^{2x} \cos 3x + 3/15 \sin 3x + 3/5 \cos 3x + 1/5 \sin 3x}$$

General Solution

Q No (4)

$$(D^2-DD')z = \cos x \cos 2y.$$

Solution:

It is already in Symbolic form.

$$(D^2-DD')z = \cos x \cos 2y \text{ --- } (a)$$

Put A.E $D^2-DD'=0$

As we know

$\frac{D}{D'} = m$ i.e $D=m, D'=1.$

$\Rightarrow$ $m^2-m=0.$

$m=0, 1.$

therefore $C.F = f_1(y)+f_2(y+x)$

From eq (a).

$$P.I = \frac{1}{D^2-DD'}\cos x \cos 2y$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-DD'} 2\cos x \cos 2y$$

As

$$2\cos A \cos B = \cos(A+B)+\cos(A-B)$$

(a) Page (13)

$\therefore$ $CF = f_1(y-x) + x f_2(y-x)$

$$PI = \frac{1}{D^2+2DD'+D'^2}\left[2(y-x)+\sin(x-y)\right]$$

$$= \frac{1}{(D+D')^2}\left[2(y-x)+\sin(x-y)\right]$$

By General Method

$m = -1 ;\ y - x = c$

$$= \frac{1}{D+D'}\int\left[2c+\sin(-c)\right]dx$$

$$= \frac{1}{D+D'}\left[2cx-(\sin c)x\right]$$

Replacing c by $y-x$

$$= \frac{1}{D+D'}\left[2x(y-x)-x\sin(y-x)\right]$$

Again Put $y-x=c$.

$$= \int(2xc - x\sin c)\,dx \Rightarrow cx^2 - \frac{x^2}{2}\sin c.$$

Replacing c by $y-x$ ~~to~~.

$$= x^2(y-x) - \frac{x^2}{2}\sin(y-x) = x^2y - x^3 + \frac{x^2}{2}\sin(x-y)$$

Hence the required solution is

$$Z = C.F + P.I = f_1(y-x) + x f_2(y-x) + x^2 y - x^3 + \frac{1}{2} x^2 \sin(x-y)$$