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Subject # Differentional Equation

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Question # (1

① $w = \sin(x+ct) + \cos(2x+2ct)$

② $w = \tan(2x+ct)$

Sol:

$$w = \sin(x+ct)\,\cos(2x+2ct)$$

$$\Rightarrow \frac{dw}{dt} = \cancel{\sin}\ \cos(x+ct)\cdot c - \sin(2x+2ct)\cdot 2c$$

$$\Rightarrow \frac{dw}{dt} = \cos(x+ct) - 2c\sin(2x+2ct)$$

$$\Rightarrow \frac{d^2w}{dt^2} = -c^2\sin(x+ct) - 2c\cos(2x+2ct)\cdot 2c$$

$$\boxed{\Rightarrow \frac{d^2w}{dt^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)} \rightarrow ①$$

$$\Rightarrow \frac{dw}{dx} = \cos(x+ct) - \sin(2x+2ct)\cdot 2$$

$$\Rightarrow \frac{d^2w}{dx^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$\Rightarrow c^2\frac{d^2w}{dx^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct) \rightarrow ②$$

Combine eq ① & eq ②

$$\Rightarrow \frac{d^2w}{dt^2} = c^2 \frac{d^2w}{dx^2}$$

Q # 1 (b)

$$\frac{d^2w}{dt^2} = c^2 \frac{d'w}{dx^2}$$

$W = Tan\,(2x + ct)$

$$\frac{dw}{dt} = Sec^2(2x + ct)\,\frac{d}{dt}(2x + ct)$$

$$= C\,Sec^2\,(2x + ct)$$

$$\Rightarrow \frac{d'w}{dt} = c^2\,Sec\,(2x + ct)\,\frac{d}{dt}$$

$$Sec\,(2x + 2\,ct)$$

$$= 2c^2\,Sec(2x + ct)\,Sec\,(2x + ct)\,Tan(2x + ct)$$

$$\Rightarrow \frac{d^2w}{dt^2} = 2C^2 Sec^2\,(2x + ct)\,Tan\,(2x + ct)$$

$$\Rightarrow \frac{dw}{dx} = 2\,Sec^2\,(2x + ct)$$

$$\frac{d^2w}{dx^2} = 2.2\sec(2x+ct).\sec(2x+ct).\tan(2x+ct).2$$

$$= 8\sec^2(2x+ct)\tan(dx+ct)$$

$$= 2c^2\sec^2(2x+ct)\tan(2x+ct)$$

$$\neq c^2 8\sec^2(2x+ct).\tan(2x+ct)$$

$\Rightarrow$ Not satisfied.

$\longleftrightarrow$

Question #2

Expand the following function in a Fourier series.

Given data:-

$$F(x) = x, \quad -\pi < x \le 0$$
$$= 2x, \quad 0 \le x \le \pi$$

Solution:-

We have to find the Fourier co-efficients, a_0, a_n & b_n.

Now:-

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[0 - \frac{\pi^{\cancel{2}}}{2}\right] + \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{2}}}{\cancel{2}} - 0\right]$$

$$\boxed{a_0 = \frac{-\pi}{2} + \pi = \frac{\pi}{2}} \longrightarrow (1)$$

$$a_n = \frac{1}{\pi}\int_{-\lambda}^{\pi} F(x)\ \cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\lambda}^{0} (x \cos nx\, dx) + \frac{1}{\pi}\int_{0}^{\pi} (2x \cos nx)\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\lambda}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos nx}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos nx}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

So

$$\boxed{a_n = \begin{cases} \frac{-2}{\pi n^2}; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases}} \longrightarrow (2)$$

$$bx = \frac{1}{\pi}\int_{-\lambda}^{\pi} F(x)\ \sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\lambda}^{0} x \sin nx\, dx + \frac{2}{\lambda}\int_{0}^{\lambda} x\ \sin nx\, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$+ \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-\frac{1 \cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-1 \cos nx}{n}\right] \rightarrow (3)$$

$$= -\frac{3 \cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}$$

So the required Function Series is :-

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$= \frac{1}{4} - \frac{2}{\lambda}\sum_{n=1}^{\infty}\frac{\cos(2x-1)x}{(2n-1)^2} + B$$

$$3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} \ldots$$

Question #13

Solve the initial value Problem.

$y'' - 4y' + 13y = 8\sin 3x,$

$y(0) = 1$ and $y'(0) = 2$

Sol:.

$$y'' - 4y' + 13y = 8\sin 3x \rightarrow ① \quad y(0) = 1, \; y'(0) = 2$$

Associated homogenous Eq of ① is

$$y'' - 4y' + 13y = 0 \rightarrow ②$$

Change ② into Auxiliary equation

Put $y = m$ in ②

$$m^2 - 4m + 13 = 0$$

Use Quardatic formula

$a = 1$, $b = -4$ $c = 13$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36i}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

~~$m_1 = 2 \pm$~~

$$\boxed{\begin{aligned} m_1 &= 2 + 3i \\ m_2 &= 2 - 3i \end{aligned}}$$

$$y_c = e^{2x}\left(c_1 \cos 3x + c_2 \sin 3x\right) \rightarrow (A)$$

Let

$$y_p = A \cos 3x + B \sin 3x \longrightarrow (*)$$

Diff w.r.to "x".

$$y'p = -3A \sin 3x + 3B \cos 3x$$

Again Diff w.r.t "x"

$$y''p = -9A \cos 3x - 9B \sin 3x$$

Put in eq (1)

$\Rightarrow (-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x$

$+ 3B\cos 3x) + 13(A\cos 3x + B\sin 3x) = 8\sin 3x$

$\Rightarrow -9A\cos 3x - 12B\cos 3x + 13A\cos 3x - 9B\sin 3x$

$+ 12A\sin 3x + 13B\sin 3x = 8\sin 3x$

$\Rightarrow (-9A - 12B + 13A)\cos 3x + (-9B$

$+ 12A + 13B)\sin 3x = 8\sin 3x$

$\Rightarrow (4A - 12B)\cos 3x + (4B + 12A)$

$\sin 3x = 8\sin 3x$

Comparing Co-efficient

$\sin 3x \Rightarrow \quad 4B + 12A = 8 \rightarrow$ (a)

$\cos 3x \Rightarrow \quad 4A - 12B = 0 \Rightarrow 4A = 12B$

$\Rightarrow \boxed{A = 3B} \rightarrow$ (b)

Put (b) in (a)

$4B + 12(3B) = 8$

$4B + 36B = 8$

$40B = 8$

$\boxed{B = 1/5} \longrightarrow$ (c)

Put (c) in (b)

$\Rightarrow \boxed{A = 3/5} \longrightarrow$ (d)

Put (c) and (d) in (*)

$$y_p = \frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \longrightarrow (B)$$

The G. Sol is

$$y = y_c + y_p$$

$$y = e^{2x} \left(c_1 \cos 3x + c_2 \sin 3x \right) + \frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \longrightarrow (c)$$

Now we need to find the values of c_1 and c_2 for this

Put $x = 0$ & $y = 1$ in (c)

$$\Rightarrow 1 = e^{x(0)} \left(c_1 \cos 3(0) + c_2 \sin 3(0) \right) + \frac{3}{5} \cos 3(0) + \frac{1}{5} \sin 3(0)$$

$$\Rightarrow 1 = \left(c_1 (1) + c_2 (0) \right) + \frac{3}{5} (1) + \frac{1}{5} (0)$$

$\Rightarrow 1 = C_1 + 3/5$

$C_1 = 1 - \frac{3}{5}$

$\boxed{C_1 = 2/5} \longrightarrow$ ★★

Diff (c) w.r.t x

$y' = C_1 (2e^{2x} \cos 3x - 3e^{2x} \sin 3x) +$

$C_2 (2e^{2x} \sin 3x + 3e^{2x} \cos 3x)$

$-\frac{6}{5} \sin 3x + \frac{3}{5} \cos 3x \longrightarrow$ (D)

Put $y' = 2$, $x = 0$ in (D)

$2 = C_1 (2e^{2(0)} \sin 3(0) + 3e^{2(0)} \sin 3(0))$
$+ C_2 (2e^{2(0)} \sin 3(0) + \frac{3}{5} \cos 3(0)$

$\Rightarrow 2 = c_1 (2) + C_2 (3) - 0 - 1 \frac{3}{5}$

$2 = 2c_1 + 3C_2 + \frac{3}{5}$

Put $c_1 = 2/5$

$2 = \frac{4}{5} + 3C_2 + \frac{3}{5}$

$2 = \frac{7}{5} + 3C_2$

$$3C_2 = 2 - \frac{7}{5}$$

$$3C_2 = \frac{3}{5}$$

$$C_2 = 3/15 \rightarrow \text{☆☆☆}$$

Put ☆☆ & ☆☆☆ in ©

$$y = e^{2x}\left(\frac{2}{5}\cos 3x + \frac{3}{15}\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$$

$$\left(y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \right)$$

Required General Solution.

Question #4

Solve

$$(D^2 - DD')\,z = \cos x \cos 2y$$

Sol:-

$$(D^2 - DD')\,z = \cos x \cos 2y$$

the given PDE can be rewrite as:

$$D(D - D')u = \cos x \cos 2y$$

in C.F is given by:

$$CF = \phi_1(y) + \phi_2(y + x)$$

while its PI is given by:

$$PI = \frac{1}{(D^2 - DD')} \cdot \frac{1}{2}\Big[\cos(x + 2y)$$

$$+ \cos(x - 2y)$$

$$= \frac{1}{2}\Big[\frac{1}{(-1+2)}\cos(x + 2y) + \frac{1}{(-1-2)}$$

$$\cos(x - 2y)\Big]$$

$$= \frac{1}{2}\cos(x + 2y) - 1/6\cos(x - 2y)$$

Hence the Complete Solution of the given PDE is given by

$$u = \phi_1(y) + \phi_2(y + x) + \frac{1}{2}\cos(x + 2y)$$
$$-1/6\cos(x - 2y).$$

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