# Final Term Paper

Name:- Hasnat Mufti

ID:- 7952

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Deptt:- BE(C)

Submitted to:- Ma'am Shomaila Mazhar

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# Question #01

The wave Equation:

We generally visit beach and ............... by the one-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

Where w is ....... propagated

Show the following functions are solutions of wave equation by determining relevant partial deriatives.

i.) $w = \sin(x+ct) + \cos(2x+2ct)$

ii.) $w = \tan(2x+ct)$

Sol:-

i) $w = \sin(x+ct) + \cos(2x+2ct)$

Sol:-

$$w = \sin(x+ct) + \cos(2x+2ct)$$

Now,

$$\frac{\partial w}{\partial t} = \cos(x+ct)+c - \sin(2x+2ct) + 2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct)+c^2 - \cos(2x+2ct) + 4c^2 \rightarrow (i)$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct) + 2$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = [-\sin(x+ct) - 4\cos(x+2ct)]$$

$$\frac{\partial^2 w}{\partial t^2} = c^2[-\sin(x+ct) - 4(\cos(2x+2ct)]$$

Hence, $c^2 \cdot \frac{\partial^2 w}{\partial x}$

$$\frac{\partial^2 w}{\partial t^2} = c^2 \cdot \frac{\partial^2 w}{\partial x^2}$$

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ii) $w = \tan(2x+ct)$

Sol:-

As we know that

$w = \tan(2x+ct)$

Taking diff w.r.t "t"

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t} \tan(2x+ct) \cdot c$$

$$\frac{\partial w}{\partial t} = \sec^2(2x+ct) \cdot c$$

$$\frac{\partial w}{\partial t} = c \sec^2(2x+ct)$$

Again taking diff

$$\Rightarrow \frac{\partial^2 w}{\partial t^2} = c \frac{\partial}{\partial t} \sec^2(2x+ct)$$

$$\Rightarrow \frac{\partial^2 w}{\partial t^2} = c \cdot 2 \sec(2x+ct) \cdot \sec(2x+ct) \cdot \tan(2x+ct) c$$

$$\frac{\partial^2 w}{\partial t^2} = 2c^2 \sec^2(2x+ct) - \tan(2x+ct)$$

Now

$$w = \tan(2x+ct)$$

Taking diff w.r.t $x$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \tan 2x+ct$$

$$\frac{\partial w}{\partial x} = \sec^2(2x+ct) \cdot 2$$

$$\frac{\partial w}{\partial x} = 2 \sec^2(2x+ct)$$

Again taking diff

$$\frac{\partial^2 w}{\partial x^2} = 2 \cdot 2 \sec(2x+ct) \sec(2x+ct) \tan(2x+ct) \cdot 2$$

$$\frac{\partial^2 w}{\partial x^2} = 6\sec^2(2x+ct).\tan(2x+ct)$$

As we know the wave equation,

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

putting values in wave equation

$$1 \neq 3$$

So we have concluded that

$$L.H.S \neq R.H.S$$

Hence, it is not the proof of wave equation,

# Question # 02

Expand the following function in a Fourier series

$$F(x) = x, \quad -\pi < x \leq 0$$
$$= 2x, \quad 0 \leq x \leq \pi$$

**Sol:-**

Given function

$$F(x) = \begin{cases} x \; ; & -\pi < x \leq 0 \\ 2x \; ; & 0 \leq x \leq \pi \end{cases}$$

we have to find the fourier co-efficients, $a_0$, $a_n$ & $b_n$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} x\,dx$$
$$+ \frac{1}{\pi} \int_{0}^{\pi} 2x\,dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$a_0 = \frac{1}{\cancel{\pi}}\left[0 - \frac{\pi^{\cancel{2}}}{2}\right] + \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^2}{\cancel{2}} - 0\right]$$

$$a_0 = -\frac{\pi}{2} + \pi = \frac{\pi}{2} \rightarrow (i)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\, \text{Cos}\, nx\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(x\, \text{Cos}\, nx\, dx) + \frac{1}{\pi}\int_0^{\pi}$$

$$(2x\, \text{Cos}\, nx)\, dx$$

$$a_n = \frac{1}{\pi}\left[x\left(\frac{\text{Sin}\, nx}{n}\right) - \left(\frac{-\text{Cos}\, nx}{n^2}\right)\right|_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\text{Sin}\, nx}{n}\right) - \left(\frac{-\text{Cos}\, nx}{n^2}\right)\right|_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\text{Cos}(0)}{n^2} - \frac{\text{Cos}\, nx}{n^2}\right] +$$

$$2/\pi\left[\frac{\text{Cos}\, nx}{n^2} - \frac{\text{Cos}(0)}{n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1-(-1)^n+2(-1)^n-2}{n^2}\right]$$

So

$$\boxed{a_n = \begin{cases} \frac{-2}{\pi n^2} & ; \text{if } n \text{ is odd} \\ 0 & ; \text{if } n \text{ is even} \end{cases}}$$

$\rightarrow$ ii

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right]$$

$$b_n = \frac{-3\cos n\pi}{n}$$

$$b_n = \frac{3(-1)^{n+1}}{n}$$

So the required fourier series is :

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2x-1)x}{(2x-1)^2} +$$

$$3 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$$

Ans

# Question # 03

Solve the initial value problem

$y'' - 4y' + 13y = 8\sin 3x$,

$y(0) = 1$ and $y'(0) = 2$

Sol:-

$$y'' - 4y' + 13y = 8\sin 3x \rightarrow (i)$$

Associated homogenous eq (i) is

$$y'' - 4y' + 13y = 0 \rightarrow (ii)$$

Change (ii) into Auxiliary equation

put $y = m$ in (ii)

$$m^2 - 4m + 13 = 0$$

use quadratic formula,

$a = 1, \ b = -4, \ c = 13$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) + \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$m = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$m = \frac{4 \pm \sqrt{36}}{2}$$

$$m = \frac{4 \pm 6i}{2}$$

$$m = 2 \pm 3i$$

$m_1 = 2 + 3i$

$m_2 = 2 - 3i$

$$y_c = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$$

A

let $y_p = A\cos 3x + B\sin 3x \rightarrow *$

Diff w.r.t $x$

$$y'_p = -3A\sin 3x + 3D\cos 3x$$

Again diff

$$y''_p = -9A\cos 3x - 9B\sin 3x$$

put in (i)

$$\Rightarrow (-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x + 3\cancel{B}D\cos 3x) + 13\,\cancel{B}(A\cos 3x + B\sin 3x) = 8\sin 3x$$

$$\Rightarrow -9A\cos 3x - 12B\cos 3x + 13A\cos 3x - 9B\sin 3x + (12A\sin 3x + 13B\sin 3x = 8\sin 3x$$

$$\Rightarrow (-9A - 12B + 13A)\cos 3x + (-9B + 12A + 12B)\sin 3x = 8\sin 3x$$

$\Rightarrow (4A-12B)\cos 3x + (4B+12A)$

$\sin 3x = 8\sin 3x$

Comparing coefficients

$\sin 3x \Rightarrow \quad 4B+12A=8 \rightarrow$ (a)

$\cos 3x \Rightarrow \quad 4A-12B=0$

$4A=12B$

$A=3B \rightarrow$ (b)

put b in a

$4B+12(3B)=8$

$4B+36B=8$

$40B=8$

$B=1/5 \rightarrow$ (c)

put c in b

$A=3/5 \rightarrow$ (d)

put cond. d in ⊛

$$y_p = \frac{B}{5} \cos 3x + \frac{1}{5} \sin 3x \rightarrow (B)$$

General Solution is:

$$y = y_c + y_p$$

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \rightarrow (C)$$

Now we need to find the values of $c_1$ and $c_2$ for this

put $x = 0$, $y = 1$ in (C)

$$1 = e^0 (c_1 \cos 3(0) + c_2 \sin 3(0)) + 3/5 \cos 3(0) + \frac{1}{5} \sin 3(0)$$

$$1 = (c_1(1) + c_2(0)) + 3/5(1) + \frac{1}{5}(0)$$

$$1 = c_1 + 3/5$$

$$c_1 = 1 - 3/5$$

$$c_1 = 2/5 \rightarrow (**)$$

Diff C w.r.t x

$$y' = c_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + c_2(2e^{2x}\sin 3x + 3e^{2x}\cos x) - 6/5 + 3/5\cos 3x \rightarrow D$$

put $y' = 2$, $x = 0$ in D

$$y' = c_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + c_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) - 6/5\sin 3x + 3/5\cos 3x$$

Now, putting values in D

$$2 = c_1(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0)) + c_2(2e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0)) - 6/5\sin 3(0) + 3/5\cos 3(0)$$

$$2 = c_1(2) + c_2(3) - 0 + 3/5$$

$$2 = 2c_1 + 3c_2 + 3/5$$

put $c_1 = 2/5$

$$2 = 4/5 + 3C_2 + 3/5$$

$$2 = 7/5 + 3C_2$$

$$3C_2 = 2 - 7/5$$

$$3C_2 = 3/5$$

$$C_2 = 3/15 \rightarrow (***)$$

put (**) and (***) in (c)

$$y = e^{2x}\left(2/5 \cos 3x + 3/15 \sin 3x\right) + 3/5 \cos 3x + 1/5 \sin 3x$$

$$y = \frac{2}{5} e^{2x} \cos 3x + 3/15\, e^{2x} \sin 3x + 3/5 \cos 3x + 1/5 \sin 3x$$

This is required general Solution,

# Question # 04

$$(D^2 - DD')z = \cos x \cos 2y$$

**Sol:-**

$$(D^2 - DD')z = \cos x \cos 2y$$

The given PDE can be rewrite as,

$$D(D - D')z = \cos x \cos 2y$$

in CF is given by:

$$CF = \phi_1(y) + \phi_2(y + x)$$

While its PI is given by:

$$PI = \frac{1}{(D^2 - DD')} \cdot \frac{1}{2}\left[\cos(x+2y) + \cos(x-2y)\right]$$

$$PI = \frac{1}{2}\left[\frac{1}{(-1+2)}\cos(x+2y) + \frac{1}{(-1-2)}\cos(x-2y)\right]$$

$$PI = \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y).$$

Hence, the complete Solution of the given PDE is given by

$$z = \phi_1(y) + \phi_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Ans