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SECTION: B

DEPARTMENT: BE Civil

SUBJECT: Differential equation

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Assignment # 02

CAUCHY EULER METHOD

Ques :- 01:

$$x^3y''' + 2x^2y' + 2y = 10x + \frac{10}{x}$$

Solution:

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{dy}{dx} + 2y = 10x + \frac{10}{x}$$

$$x^3D^3y + 2x^2D^2 + 2y = 10x + 10x^{-1}$$

$$y(x^3D^3 + 2x^2D + 2)y = 10x + 10x^{-1} \text{ ——— } (1)$$

Let; $x = e^t \implies t = \ln x$

$$xD = D$$

$$x^2D^2 = D(D-1) = D^2 - D$$

$$x^3D^3 = D(D-1)(D-2)$$

substituting into eq (1)

$$(D - 3D^2 + 2D + 2(D^2 - D) + 2)y = 10^x + 10x^{-1}$$

$$(D^3 - D^2 + 2)y = 10x + 10x^{-1}$$

$$(m^3 - m^2 + 2)y = 10e^t + \frac{10}{e^t}$$

using synthetic devision.

-1	1	-1	0	2
		-1	2	-2
	1	-2	2	0

$$D^2 - 2D + 2 = 0$$

By using quadratic formula:

$a = 1$, $b = -2$ and $c = 2$.

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$D = \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-1} \cdot \sqrt{4}}{2}$$

$$D = \frac{2 + 2i}{2}$$

$$D = \frac{\cancel{2}(1 + i)}{\cancel{2}}$$

$$D = 1 \pm i$$

Since root are complex.

$$y_c = e^{-x}(c_1 \cos t + C_2 \sin t).$$

Now particular integration.

$$y_p = \frac{1}{D^3 - D^2 + 2} \cdot 10e^t + \frac{1}{D^3 - D^2 + 2} \cdot 10/e^t.$$

$$= \frac{10e^t}{(1)^3 - (1)^2 + 2} + \frac{10e^{-t}}{(1)^3 - (1)^2 + 2}$$

$$= \frac{10e^t}{2} + \frac{10e^{-t}}{2}$$

$$= 5e^t + 5e^{-t}$$

$$y_p = 5e^t + 5e^{-t}$$

⇒ General Solution:

$$y = y_c + y_p$$

$$y = e^{-x}(C_1 \cos t + C_2 \sin t) + 5e^t + 5e^{-t}$$

Put $e^t = x$ and $t = \ln x$

$$y = e^{-x}(C_1 \ln x + C_2 \sinh x) + 5e^x + 5e^{-x}$$ Ans

Ques : No #02 :

$$x^3 \frac{d^3y}{dx} + 4x^2 \frac{d^2y}{dx} - 5x \frac{dy}{dx} - 15y = x^4$$

Solution:

let $d/dx = D$

$$x^3 D^3 y + 4x^2 D^2 y - 5x Dy - 15y = x^4$$

$$(x^3 D^3 + 4x^2 D^2 - 5x D - 15)y = x^4$$

Let; $x = e^t \Rightarrow t = \ln x$.

$$xD = D.$$

$$x^2 D^2 = D(D-1) = D^2 - D.$$

$$x^3 D^3 = D(D-1)(D-2) = D^3 - 3D^2 + 2D$$

Now substituting :-

$$(x^3 D^3 + 4x^2 D^2 - 5xD - 15)y = x^4$$

$$\left[D^3 - 3D^2 + 2D + 4(D^2 - D) - 5(D) - 15\right]y = e^{4t}$$

$$(D^3 + D^2 - 7D - 15)y = e^{4t}$$

By Synthetic devision.

$$
\begin{array}{c|cccc}
 & 1 & +1 & -7 & -15 \\
5 & & 3 & 12 & 15 \\
\hline
 & 1 & 4 & 5 & \boxed{0}
\end{array}
$$

$$D^2 + 4D + 5 = 0.$$

Quadratic formula.

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here; $a = 1,\ b = 4,\ c = 5.$

$$D = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)}.$$

$$D = \frac{-4 \pm \sqrt{16-20}}{2}.$$

$$D = \frac{-4 \pm \sqrt{-4}}{2}.$$

$$D = \frac{-4 \pm \sqrt{-1} \cdot \sqrt{4}}{2}$$

$$D = \frac{-4 \pm 2i}{2}.$$

$$D = -2 \pm i$$

$y_c = e^{3x}(c_1 \cos t + c_2 \sin t)$

For $y_p = ?$

$$y_p = \frac{1}{D^3 + D^2 - 7D - 15} \cdot e^{4t}$$

$$y_p = \frac{1}{(4)^3 + (4)^2 - 7(4) - 15} \cdot e^{4t}$$

$$y_p = \frac{1}{64 + 16 - 28 - 15} \cdot e^{4t}$$

$$y_p = \frac{1}{80 - 43} \cdot e^{4t}$$

$y_p = 1/37 . e^{4t}$

Now; $y = y_c + y_p$.

$y = (c_1 \cos t + c_2 \sin t) + 1/37\, e^{4t}$

again put $t = \ln x$ & $x = \ln x$.

$y = e^{3x}(c_1 \cos \ln x + c_2 \sin \log u) + 1/37\, e^{4x}$.

Ans:

Ques . No: # 03:

$x^2 y'' + 2xy' - 6y = 10x^2$; $y(1) = 1$, $y'(1) = 6$

Solution:-

$x^2 y'' + 2xy' - 6y = 10x^2$

$y(1) = 1$ and $y'(1) = 6$.

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2$$

$$\Rightarrow (x^2 \, d^2/dx^2 + 2x \, d/dx - 6)y = 10x^2$$

put $xD = D \Rightarrow x^2 D^2 = D(D-1) = D^2 - D$

$x = e^t$ and $\log x = t$.

$(D^2 + 2D - D - 6)y = 10e^{2t}$

$(D^2 + D - 6)y = 10e^{2t}$

The characteristic equation.

$D^2 + D - 6 = 0$

$D^2 + 3D - 2D - 6 = 0$

$\Rightarrow D(D+3) - 2(D+3) = 0$

$(D-2)(D+3) = 0$

$D = 2$, $D = -3$.

Since root are real and distinct.

For $y_c = ?$

$$y_c = C_1 e^{-3t} + C_2 e^{2t}$$

For $y_p = ?$

$$y_p = \frac{1}{D^2 - D - 6} \cdot 10 e^{2t}$$

$$y_p = \frac{10}{D^2 - D - 6} e^{2t}$$

$$= 10 \frac{1}{0} e^{2t}$$

Now,

$$10 \frac{1}{d/dD(D^2 + D - 6)} e^{2t}$$

$$\Rightarrow 10 \frac{t}{2D+1} e^{2t}$$

$$= 10 \frac{1 - t}{4+1} e^{2t}$$

$$y_p = 2te^{2t}.$$

General Solution

$$y = y_c + y_p$$

$y = C_1 e^{-3t} + C_2 e^{2t} + 2t e^{2t}$

$y = C_1 x^{-3} + C_2 x^2 + 2(\log x) x^2$ ——— (B)

put $y(1) = 1$ i.e $x = 1$ & $y = 1$ in (B)

$1 = C_1 (1)^{-3} + C_2 (1)^2 + 2(\log(1)) (1)^2$

$1 = C_1 + C_2$ ——— (C)

Now differentiate eq (B) w.r.t x.

$y' = -3C_1 x^{-4} + 2C_2 x + \frac{2}{x}(x^2) + 4x \log x.$

Now put $y'(1) = -6$ i.e $y' = -6$ & $x = -6$

$-6 = -3C_1 + 2C_2 + 2 + 0.$

$\Rightarrow -6 = -3C_1 + 2C_2 + 2.$

$\Rightarrow -6 - 2 = -3C_1 + 2C_2.$

$-8 = -3C_1 + 2C_2$ ——— (D)

X-ing eq (C) with (2) and -ing from (D)

$$\begin{array}{rcl} 2C_1 + 2C_2 & = & 2 \\ \mp 3C_1 \pm 2C_2 & = & \mp 8 \\ \hline 5C_1 & = & 10 \end{array}$$

$5C_1 = 10.$

$C_1 = 10/5 \boxed{= 2.}$

$-8 = -3(2) + 2(C_2).$

$-8 = -6 + 2C_2.$

$-8 + 6 = 2C_2$

$-2 = 2C_2.$

$\boxed{C_2 = -1.}$

Now put this values of C_1 & C_2 in (B)

$y = 2x^{-3} - x^2 + 2\ln x \; x(x^2).$

$y = \frac{2}{x^3} - x^2 + 2x^2 \log x$ Ans:-

Ques : No # 04:

$$x^2 y'' + 7xy' + 5y = x^5$$

$y(0) = 2$ & $y'(1) = 2$

Solution:-

$$x^2 \frac{dy^2}{dx^2} + 7x\frac{dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2 \frac{d^2}{dx} + 7x\frac{d}{dx} + 5\right) y = x^5 \quad \text{———} \ (A)$$

put $xD = D \Rightarrow x^2D^2 = D(D-1) = D^2 - D$

$x = e^t \Rightarrow \log x = t$ in eq (A)

$$\Rightarrow (D^2 - D + 7D + 5)y = e^{5t}$$

$$\Rightarrow (D^2 + 6D + 5)y = e^{5t}$$

By Quadratic formula:-

$$D = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$D = \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$D = \frac{-6 \pm \sqrt{16}}{2}.$$

$$D = \frac{-6 \pm 4}{2}.$$

$$D = \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}}$$

$$D = -3 \pm 2.$$

Since roots are real & distinct.

$$y_c = C_1 e^{-5t} + C_2 e^{-t}$$

For $y_p = ?$

$$y_p = \frac{1}{D^2 + 6D + 5} e^{5t}$$

$$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$$

$$= \frac{1}{60} e^{5t}$$

Now general solution is

$$y = y_c + y_p.$$

$$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t}$$

$$y = C_1 x^{-5} + C_2 x^{-1} + \frac{1}{60} x^5 \longrightarrow (B)$$

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$x = 0$ put in this eq B
$e^0 = 1$.

Put $y(0) = 2$ i-e $y = 2$ & $x = 2$.

$$2 = c_1 (2)^{-5} + c_2 (2)^{-1} + \frac{1}{60} (2)^5$$

$$2 = -32c_1 - 2c_2 + \frac{1}{60} (30)$$

$$2 = -32c_1 - 2c_2 + \frac{8}{18}.$$

$$2 - \frac{8}{15} = -32c_1 - 2c_2.$$

$$22/15 = -32c_1 - 2c_2 \quad \text{——— (C)}$$

Now differentiating eq (B) w.r.t x.

$$y' = -5c_1 x^{-6} - c_2 x^{-2} + \frac{1}{12} x^4$$

put $y'(1) = 2$ i-e $y' = 2$ & $x = 2$
in above eq.

$$\cancel{2 = -5c_1 c_2 - 8 - c_2}$$

$$2 = -5c_1 (2)^{-6} - c_2 (2)^{-2} + \frac{1}{12} (2)^4$$

$$2 = -5c_1 (64) - c_2 (4) + \frac{1}{12} (16)$$

$$2 = 320c_1 + 4c_2 + 4/3$$

$2 - 4/3 = 320C_1 + 4C_2$

$2/3 = 320C_1 + 4C_2$ —— (D)

x-ing eq (C) with (2) and then ÷ing eq (C) from (D).

$\frac{-44}{15} = 64C_1 + 4C_2$

$\frac{-44}{15} = 64C_1 + 4C_2$

$+2/3 = \pm 320C_1 \pm 4C_2$

$\frac{34}{15} = -256C_1$

$\boxed{C_1 = 580}$

put the value of C_1 in eq (C)

$\frac{22}{15} = -18560 - 2C_2$

$\Rightarrow \frac{22}{15} + 18560 = -2C_2$

$\boxed{C_2 = -9280}$

put these values in (c)

$$\frac{22}{15} = -32(580) - 2c_2.$$

$$\frac{22}{15} = -18560 - 2c_2.$$

$$\frac{22}{15} + 18560 = -2c_2.$$

$$c_2 = \frac{18561}{-2}$$

$$\boxed{c_2 = -9280}$$

Now put the value of c_1 & c_2 in (B)

$$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$$ Ans:-

Ques : # 05 :

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$$

Solution-

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$$

$$(x+1)^2 \frac{d^2y}{dx} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$\Rightarrow \left[(x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right] y = x^2$$

$$\Rightarrow \left[(x+1)^2 D^2 - 3(x+1)D + 4\right] y = x^2 \quad \text{— (A)}$$

put $(x+1)D = D \Rightarrow (x+1)^2 D^2 = D(D-1)$
$= D^2 - D$

$x = e^t$ in eq (A)

$$\Rightarrow \left[D^2 - D - 3D + 4\right] y = e^{2t}$$

$$\Rightarrow (D^2 - 4D + 4) y = e^{2t}$$

for y_c we find the roots.

$$D^2 - 4D + 4 = 0$$
$$D^2 - 2D - 2D + 4 = 0$$
$$D(D-2) - 2(D-2) = 0$$
$$D = 2, \ D = 2.$$

So, root are real & repeat.
The general solution is;

$$y = (C_1 + C_2 x)^{mx}$$

$$y = (C_1 + C_2 x)^{2x}$$

For $yp = ?$

$$yp = \frac{1}{D^2 - 4D + 4}$$

$$yp = \frac{2}{2D - 4} e^{2t}$$

If we put "2"

$$2D - 4 \rightarrow 2(2) - 4 = 0.$$

we take again derivative.

$$yp = \frac{\cancel{2}}{\cancel{2}} e^{2t} = e^{2t}$$

$$y = (C_1 + C_2 x)^{2t} + e^{2t}$$

General Solution : Ans.