①

M☐T☐W☐T☐F☐S H/W☐ C/W☐ Dated:........./........./20........

Name :- Hayat ahmad khan

ID # 14486

Subject :- Differential Equations

Department :- Computer science

Exam :- Summer / Mid-Exams

University :- Iqra national university

Instructor :- Engr. Latif Jan

-:Q1:-

Part (a)

Differential equation:-

A differential equation is an equation that relates the unknown function to some of its derivatives, which of course, are not known either.

Example:-

$$\theta'' + \sqrt{\frac{g}{l}} \sin\theta = 0,$$

$$p' = rp\left(1 - \frac{p}{k}\right),$$

-: Q1 :-

Part (B)

Separable equation:-

A differential equation having the form

$$x' = f(x)\, g(t),$$

where the right side is a product of a function of x and a function of t. is called a separable equation.

(i) $$y' = \frac{xy^3}{\sqrt{1+x^2}} \quad y(0) = -1$$

$$y' = \frac{x y^3}{\sqrt{1+x^2}}$$

$$\frac{1}{y^3}\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$$

4

$$\frac{1}{y^3}\,dy = \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\int y^{-3}\,dy = \int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\int y^{-3}\,dy = \int x\,(1+x^2)^{-1/2}\,dx$$

$$\frac{y^{-3+1}}{-3+1} = \frac{1}{2}\int 2x\,(1+x^2)^{-1/2}\,dx$$

$$\frac{y^{-2}}{-2} = \frac{1}{2}\,\frac{(1+x^2)^{-1/2+1}}{-1/2+1} + c$$

$$\frac{-1}{2y^2} = \frac{1}{2}\,\frac{(1+x^2)^{1/2}}{1/2} + c$$

$$-1/2y^2 = \cancel{2}\cdot\frac{1}{\cancel{2}}\,(1+x^2)^{1/2} + c$$

$$\frac{-1}{2y^2} = \sqrt{1+x^2} + c$$

$$2y^2 = \frac{-1}{\sqrt{1+x^2}} - 1/c$$

Apply initial condition,

$$y^2 = -\frac{1}{2}(1+x^2)^{-1/2} - \frac{1}{c}$$

$$y = \frac{-1}{\sqrt{2}}\sqrt{(1+x^2)^{-1/2}} - \frac{1}{c}$$

$$y = \frac{-1}{\sqrt{2}}(1+x^2)^{-1/2+1/2} - \frac{1}{c}$$

$$y = -\frac{1}{\sqrt{2}}(1) - \frac{1}{c}$$

$$y = \frac{-1}{\sqrt{2}} - \frac{1}{c} \longrightarrow (A)$$

Apply initial condition

$$y(0) = -1$$

$$1 = \sqrt{2} + c$$

$c = -1$ put in (A)

$$y(x) = -\frac{1}{\sqrt{2}} - \frac{1}{1}$$

$$\boxed{y(x) = \frac{-1}{\sqrt{2}} + 1}$$

(ii)

$$\frac{dx}{dt} = \frac{t}{x}$$

$$\frac{dx}{dt} = \frac{t}{x}$$

Seperating the variables we get

$$x \frac{dx}{dt} = t$$

integrating

$$\int x \frac{dx}{dt} dt = \int t \, dt,$$

Since

$$dx = \frac{dx}{dt} dt, \therefore$$

$$\int x \, dx = \int t \, dt$$

$$\frac{1}{2} x^2 = \frac{1}{2} t^2 + c$$

## -: Q2 :-

### Part (a)

Steps for Solving Linear Differential Equation.

**Step 1 :-**

Multiply both sides of the normal form of the equation

$$x' + p(t)x = q(t)$$

by integrating factor

$$\mu(t) = e^{\int p(t)\,dt} = e^{P(t)}$$

**Step 2 :-**

Obtain

$$\left(e^{P(t)} x\right)' = e^{P(t)} q(t).$$

**Step 3 :-**

integrate both sides to obtain

$$e^{P(t)} x(t) = \int e^{P(t)} p(t)\,dt + C$$

**Step 4 :-**

Multiply by $e^{-P(t)}$ to obtain the general solution

$$x(t) = e^{-P(t)} \int e^{P(t)} p(t)\,dt + Ce^{-P(t)}$$

A (ii) $\cos(x)\, y' + \sin(x)\, y = 2\cos^3(x)\sin(x) - 1$

$$y\left[\frac{\pi}{4}\right] = 3\sqrt{2}, \quad 0 \le x \le \frac{\pi}{2}$$

$$\cos(x)\, y' + \sin(x) y = 2\cos^3(x)\, \sin(x) - 1$$

$$y' + \frac{\sin(x)}{\cos(x)} y = \frac{2\cos^3(x)\sin(x) - 1}{\cos x}$$

$$y' + \tan(x)\, y = 2\cos^2(x)\sin(x) - \sec x$$

Compare eq Ⓐ with

$y' + p(x) y = q(x)$ we obtain

$p(x) = \tan(x) \cdot q(x) = 2\cos^2(x)\sin(x) - \sec x$

The integrating factor is

$$\mu(t) = e^{\int p(x)\,dx} = e^{\int \tan x\,dx} = e^{-\ln(\cos x)}$$

$$= -\cos(x)$$

Multiply $-\cos x$ to both sides of eq Ⓐ

$$-\cos(x)\, y' - \tan(x)\cos(x)\, y = -2\cos^3(x)\sin x + \sec(x)\cos(x)$$

$$-\cos(x) y' - \sin(x) y = -2\cos^3(x)\sin(x) + 1$$

$$(-\cos(x) y)' = -2\cos^3(x)\sin(x) + 1$$

=)

Integrate both sides

$$\int (-\cos(x) y)' dx = \int -2\cos^3(x)\sin(x)dx + \int 1 dx$$

$$-\cos(x)y = -2\int \cos^3(x)\sin(x)dx + x + C$$

$$-\cos(x)y = 2\int \cos^3(x)(-\sin(x))dx + x + C$$

$$-\cos(x)y = 2\frac{\cos^{3+1}(x)}{3+1} + x + C$$

$$-\cos(x)y = \frac{1}{2}\cos^4(x) + x + c$$

$$y = -\frac{1}{2}\cos^3(x) - \frac{x}{\cos x} - \frac{c}{\cos x} \rightarrow 1$$

Apply intial Condition.

$$y\left[\frac{\pi}{4}\right] = 3\sqrt{2}$$

$$3\sqrt{2} = -\frac{1}{2}\cos^3(\pi/4) - \frac{\pi/4}{\cos(\pi/4)} - \frac{c}{\cos(\frac{\pi}{4})}$$

$$3\sqrt{2} = -\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^3 - \frac{\pi/4}{(\sqrt{2}/2)} - \frac{c}{\left(\frac{\sqrt{2}}{2}\right)}$$

$$3\sqrt{2} = -\frac{1}{\cancel{2}}\left(\frac{\cancel{2}\sqrt{2}}{8}\right) - \frac{\cancel{2}\pi}{\cancel{4}\sqrt{2}} - \frac{2c}{\sqrt{2}}$$

$$3\sqrt{2} = \frac{\sqrt{-2}}{8} - \frac{\pi}{2\sqrt{2}} - \frac{2c}{\sqrt{2}}$$

$$6 = \frac{-2}{8} - \frac{\pi}{2} - 2c$$

$$6 = -\frac{1}{4} - \frac{\pi}{2} - 2c$$

$$6 + \frac{1}{4} - \frac{\pi}{2} = -2c$$

$$\frac{24 + 1 - 2\pi}{4} = -2c$$

$$\frac{25 - 2\pi}{4} = -2c$$

$$\frac{2\pi - 25}{8} = c \quad \text{put in ①}$$

$$\boxed{y(x) = -\frac{1}{2}\cos^3(x) - \frac{x}{\cos x} - \frac{(2\pi - 25)}{8\cos x}}$$

(ii) $x' + 2x = \sin t$

$$x' + 2x = \sin t$$

we multiply the (DE) by the integrating factor

$$\mu(t) = e^{\int 2\,dt} = e^{2t}$$

$$(xe^{2t})' = e^{2t}\sin t$$

Integrating both sides gives

$$xe^{2t} = \int e^{2t}\sin t\,dt + c$$

(or)

$$x(t) = e^{-2t}\int e^{2t}\sin t\,dt + ce^{-2t}$$

$$x(t) = e^{-2t}\left[e^{2t}\left(\frac{2}{5}\sin t - \frac{1}{5}\cos t\right)\right]$$

$$+ Ce^{-2t}$$

$$= 2/5 \sin t - 1/5 \cos t + Ce^{-2t}$$

-: Q3: -

Part (i)

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = -3$$

$$2xy - 9x^2 + (2y + x^2 + 1)\, dy/dx = 0$$

$$(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$$

Here

$$M = 2xy - 9x^2$$
$$N = 2y + x^2 + 1$$

$$\frac{\partial M}{\partial y} = 2x \longrightarrow (1)$$

$$\frac{\partial N}{\partial x} = 2x \longrightarrow (2)$$

(1) = (2) equation is exact.

Now

$u = \int M\,dx + k(y)$ or $u = \int N\,dy + k(x)$

$u = \int (2xy - 9x^2)\,dx + k(y)$

$u = 2\int xy\,dx - 9\int x^2\,dx + k(y)$

$u = 2\frac{x^2}{2}y - 9\frac{x^3}{3} + k(y)$

$u = x^2y - 3x^3 + k(y) \rightarrow (1)$

diff w.r.t $y$

$\frac{du}{dy} = x^2 + \frac{dk}{dy}$

As $N = \frac{du}{dy}$

$2y + x^2 + 1 = x^2 + \frac{dk}{dy}$

$\frac{dk}{dy} = 2y + 1$

$dk = (2y + 1)\,dy$

$\int dk = 2\int y\,dy + \int 1\,dy$

$k = \frac{2y^2}{2} + y$

$k = y^2 + y + c_1$

put in (1)

$u = x^2y - 3x^3 + y^2 + y + C1$

$C_2 - C_1 = x^2y - 3x^3 + y^2 + y$

$c = (x^2+1)y + y^2 - 3x^3$

$3x^3 - y^2 + c = (x^2+1)y$

$y = \dfrac{3x^3 - y^2 + c}{x^2+1} \rightarrow (2)$

put intial condition $y(0) = -3$

$-3 = \dfrac{0 - 9 + c}{0+1}$

$-3 = -9 + c$

$\boxed{c = 6}$

put in (2)

$$\boxed{y(x) = \frac{3x^3 - y^2 + 6}{x^2+1}}$$

-: Q3 :-

Part (ii)

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0 , \quad y(5) = 0$$

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0$$

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))\frac{dy}{dt} = 0$$

$$\left(\frac{2ty}{t^2+1} - 2t\right)dt - (2 - \ln(t^2+1))dy = 0$$

Here

$$M = \frac{2ty}{t^2+1} - 2t$$

$$N = -(2 - \ln(t^2+1))$$

$$N = \ln(t^2+1) - 2$$

$$\frac{\partial M}{\partial y} = \frac{2t}{t^2+1} \rightarrow ①$$

$$\frac{\partial N}{\partial t} = \frac{1}{t^2+1}(2t) = \frac{2t}{t^2+1} \rightarrow ②$$

① = ② equation is exact

Now

$u = \int M\,dt + k(y)$ or $u = \int N\,dy + k(t)$

$u = \int \left( \frac{2ty}{t^2+1} - 2t \right) dt + k(y)$

$u = 2y \int \frac{t}{t^2+1}\,dt - 2\int t\,dt + k(y)$

$u = \frac{\cancel{2}y}{\cancel{2}} \int \frac{2t}{t^2+1}\,dt - \cancel{2}\frac{t^2}{\cancel{2}} + k(y)$

$u = y \ln(t^2+1) - t^2 + k(y) \rightarrow (1)$

diff w.r.t $y$

$\frac{du}{dy} = \ln(t^2+1) + \frac{dk}{dy}$

As $N = \frac{du}{dy}$

$\cancel{\ln(t^2+1)} - 2 = \cancel{\ln(t^2+1)} + \frac{dk}{dy}$

$\frac{dk}{dy} = -2$

$dk = -2\,dy$

$\int dk = -2\int 1\,dy$

$\boxed{k = -2y + c_1}$ put in (1)

$u = y \ln(t^2+1) - t^2 - 2y + c_1$

$$C_2 - C_1 = y \ln(t^2+1) - t^2 - 2y$$

$$c = y \ln(t^2+1) - t^2 - 2y$$

$$c + 2y = y \ln(t^2+1) - t^2$$

$$2y = y \ln(t^2+1) - t^2 - c$$

$$y = \frac{y \ln(t^2+1) - t^2 - c}{2} \rightarrow (2)$$

put initial condition; $y(5) = 0$

$$0 = \frac{0 - 5^2 - c}{2}$$

$$0 = -25 - c$$

$\boxed{c = -25}$ put in (1)

$$\boxed{y(x) = \frac{y \ln(t^2+1) - t^2 + 25}{2}}$$