THE CAUCHY - EULER ODE'S

Q.No.1: $x^3y''' + 2x^2y'' + 2y = 100x + \frac{10}{x}$

Solution:

$$x^3y''' - 2x^2y'' + 2y = 100x - \frac{10}{x} \longrightarrow Eq\ ①$$

let $x = e^t$

then $\frac{d}{dt}x = e^t \Rightarrow \frac{dt}{dx} = e^{-t}$

Now $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}e^{-t}$

$$y' = e^{-t}\Delta y \qquad \therefore d/dt = \Delta$$

Similarly $y'' = e^{-2t}[\Delta(\Delta-1)]y$

$$y''' = e^{-3t}[\Delta(\Delta-1)(\Delta-2)]y$$

Putting all these values in eq ①

$$\cancel{e^{3t}}\cancel{e^{-3t}}[\Delta(\Delta-1)(\Delta-2)]y + 2\cancel{e^{2t}}.\cancel{e^{-2t}}[\Delta(\Delta-1)]y + 2y = 100e^t + 10e^{-t}$$

$$(\Delta^3 - 3\Delta^2 + \cancel{2\Delta})y + (2\Delta^2 - \cancel{2\Delta})y + 2y = 100e^t + 10e^{-t}$$

$$\Delta^3y - \Delta^2y + 2y = 100e^t + 10e^{-t}$$

or

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 100e^t + 10e^{-t} \rightarrow ②$$

The associated Homogenious equation of eq ①

$$\Rightarrow \frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 0$$

let $\frac{d}{dt^2} = K^2$, $\frac{d^3}{dt^3} = K^3$

$$\Rightarrow (K^3y - K^2y + 2y) = 0$$

$$\Rightarrow (K^3 - K^2 + 2)\ y = 0$$

For non-trivial solution, $y \neq 0$

$$\Rightarrow K^3 - K^2 + 2 = 0$$

$$\Rightarrow K = -1,\ K = \pm 2.$$

$$y_c(t) = Ae^{-t} + (B\cos t + C\sin t)\,e^t$$

which is complementary solution.

Q.2.

$$x^3y''' + 4x^2y'' - 5xy' - 15y = x^4 \rightarrow ①$$

Sol.

Let $x = e^t$

$$\frac{d}{dt}x = e^t \Rightarrow \frac{dt}{dx} = e^{-t}$$

Now $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}e^{-t}$

$$y' = e^{-t}\Delta y \quad \therefore \frac{d}{dt} = \Delta$$

Similarly $y'' = e^{-2t}[\Delta(\Delta-1)]y$

$$y''' = e^{-3t}[\Delta(\Delta-1)(\Delta-2)]y$$

Putting all the values in Eq ①.

$$\cancel{e^{3t}}\cancel{e^{-3t}}[\Delta(\Delta-1)(\Delta-2)]y + 4\cancel{e^{2t}}\cancel{e^{-2t}}[\Delta(\Delta-1)]y - 5\cancel{e^t}\cancel{e^{-t}}\Delta y - 15y = e^{4t}$$

$$\Rightarrow (\Delta^3 - 3\Delta^2 + 2\Delta)y + 4(\Delta^2 - \Delta)y - 5\Delta y - 15 = e^{4t}$$

$$\Rightarrow \Delta^3 y + \Delta^2 y - 7\Delta y - 15y = e^{4t} \rightarrow ②$$

$$\Delta^3 + \Delta^2 - 3\Delta - 15 = 0$$

$\Rightarrow$ Roots of $\Delta = 1, 3, 5$

Complementaly eq is

$$C.F = c_1 e^{-3t} + c_2 e^{-5t}$$

$$P.I = \frac{1}{D^3 + D^2 - 3D - 15} e^{3t}$$

Putting $D = 3$

$$P.I = \frac{1}{(3)^3 - (3)^2 - 3(3) - 15} e^{3t}$$

$$P.I = \frac{1}{3} e^{3t}$$

Thus general solution

$$y = c_1 e^{3t} + c_2 x^{-t} + \frac{1}{3} e^{3t}$$

y

Q.3:

Solution:

$$x^2 y'' + 2xy' - 6y = 10x^2 \longrightarrow ①$$

$$y(1) = 1, \quad y'(1) = -6$$

Let $x = e^t$

Now $xy' = \Delta y$

$\Rightarrow x^2 y'' = \Delta(\Delta - 1) y$

$\because \Delta = \frac{d}{dt}$

So,

Eq ① $\Rightarrow$ $\{\Delta(\Delta-1) + 2\Delta - 6\} y = 10e^{2t}$

$\Rightarrow (\Delta^2 + \Delta - 6) y = 10e^{2t}$

For Δ $\Rightarrow$ $\Delta^2 + \Delta - 6 = 0$

$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$

$(\Delta + 3)(\Delta - 2) = 0$

$\Rightarrow \Delta = -3, \quad \Delta = 2$

Now

$$C.F = c_1 e^{-3t} + c_2 e^{2t}$$

Also P. Integral

$$P.I = \frac{1}{\Delta^2 + \Delta - 6} 10e^{2t}$$

$$= 10\left(\frac{1}{(2)^2 + 2 - 6} e^{2t}\right)$$

$\Rightarrow P.I = 10t \frac{1}{2\Delta + 1} e^{2t}$

$= 10t \frac{1}{5} e^{2t}$

$P.I = 2te^{2t}$

General Solution $\Rightarrow$ $y = C.F + P.I$

$$y = c_1 e^{-3t} + c_2 e^{2t} + 2t e^{2t}$$

$$y = c_1 x^{-3} + c_2 x^2 + 2(\log x) x^2 \rightarrow (*)$$

Now Applying initial Condition $y(1) = 1$

$$\Rightarrow 1 = c_1 + c_2 + 0 \rightarrow (a)$$

& $y'(1) = -6$

$$\Rightarrow -6 = -3c_1 + 2c_2 + 2 + 0$$

$$\Rightarrow -3c_1 + 2c_2 = -8 \rightarrow (b)$$

Adding (3×eq (a)) & eq (b)

$$\Rightarrow \begin{array}{l} 3 = \cancel{3c_1} + 3c_2 \\ -8 = \cancel{-3c_1} + 2c_2 \\ \hline -5 = 5c_2 \end{array}$$

$$\Rightarrow \boxed{c_2 = -1}$$

& $\boxed{c_1 = 2}$

Thus Eq (*) $\Rightarrow$ $\boxed{y = 3x^{-3} - x^2 + 2x^2 \log x}$

Final Answer.

Q.4:-

$$x^2y'' + 7xy' + 5y = x^5$$

$y(0) = 2$
$y'(1) = 2$

Let $x = e^t \Rightarrow t = \log x$

$\Delta = \frac{d}{dt}$

Now $xy' = \Delta y \Rightarrow x^2y'' = \Delta(\Delta - 1)y$

then

$$(\Delta(\Delta-1) + 7\Delta + 5)y = e^{5t}$$

Charac eq $\Delta^2 + 6\Delta + 5 = 0$

$\Rightarrow \Delta = -5, \Delta = -1$

Complementary eq is

$$C.F = c_1 e^{-5t} + c_2 e^{-t}$$

$$P.I = \frac{1}{\Delta^2 + 6\Delta + 5} e^{5t}$$

Putting $\Delta = 5$

$$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$$

$$P.I = \frac{1}{60} e^{5t}$$

Thus general solution

$$y = c_1 e^{-5t} + c_2 x^{-1} + \frac{1}{60} e^{5t} \longrightarrow (*)$$

$$y' = -5c_1 x^6 - c_2 x^{-2} + \frac{1}{12} x^4$$

$y = (0) = 2$, $x = 0$, $y = 2$

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$$2 = c_1 + c_2 + \frac{1}{60}$$

$$c_1 + c_2 = \frac{119}{60} \rightarrow (a)$$

$y'(1) = 2, \quad x = 1, \quad y' = 2$

$$2 = -5c_1 - c_2 + \frac{1}{12}$$

$$5c_1 - c_2 = \frac{23}{12} \rightarrow (b)$$

Adding (a) & (b)

$$-4c_1 = \frac{234}{60}$$

$$\Rightarrow c_1 = \frac{117}{120}$$

/ & Now $c_2 = \frac{355}{120}$

So eq (*)

$$\Rightarrow y = \frac{117}{120} e^{-5t} + \frac{355}{120} x^{-1} + \frac{1}{60} e^{5t}$$

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Q.5:

Solution:

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2 \rightarrow (1)$$

Let $x+1 = e^t \Rightarrow x = e^t - 1$

$\log(x+1) = t$

&

$(x+1)y' = \Delta y \quad , \quad \Delta = \frac{d}{dt}$ & Δ

$(x+1)^2 y'' = \Delta(\Delta-1)y$

So Eq (1) $\Rightarrow$ $(\Delta(\Delta-1) - 3\Delta + 4)\,y = (e^t - 1)^2$

$(\Delta^2 - 4\Delta + 4)\,y = e^{2t} - 2e^t + 1$

For $\Delta \Rightarrow$ $\Delta^2 - 4\Delta + 4 = 0$

$(\Delta-2)^2 = 0$

$\Delta = 2, \Delta = 2$

So

Complementary function $= (c_1 + c_2 t)\,e^{2t}$

& $P.I = \frac{1}{(\Delta-2)^2}(e^{2t} - 2e^t + 1)$

$$P.I = \frac{1}{(\Delta-2)^2} e^{2t} - \frac{2e^t}{(\Delta-2)^2} + \frac{1}{(\Delta-2)^2} \rightarrow (2)$$

So

$$\frac{1}{(\Delta-2)^2} e^{2t} = \frac{1}{(2-2)^2} e^{2t}$$

$$= \frac{1}{0} e^{2t}$$

Case of failure

$$\frac{1}{(\Delta-2)^2}e^{2t} = \frac{1}{(1-2)^2}e^{2t}$$

$$= \frac{t^2}{2}\left(\frac{1}{1}\right)e^t$$

$$= \frac{t^2e^t}{2}$$

Ee $\quad 2\frac{1}{(\Delta-2)^2}e^t = 2\frac{1}{(1-2)^2}e^t = 2e^t$

Ee $\quad \frac{1}{(\Delta-2)^2}(1) = \frac{1}{(\Delta-2)^2}e^{0t} = \frac{1}{4}$

Now eq ② $\Rightarrow \quad P.I = \frac{1}{2}t^2e^{2t} - 2e^t + \frac{1}{4}$

Hence Complete Solution is

$$y = C.f + P.I$$

$$y = (c_1 + c_2t)e^{2t} + \frac{1}{2}t^2e^{2t} - 2e^t + \frac{1}{4}$$

Putting value of e^{t}

$$y = (c_1 + c_2\log(x+1))(x+1)^2 + \frac{1}{2}\{\log(x+1)^2(x+1)^2\} - 2(x+1) + \frac{1}{4}$$

$$y = (c_1 + c_2\log(x+1))(x+1)^2 + \frac{1}{2}\{\log(x+1)^2(x+1)^2\} - 2x - 7/4$$

Answer.