Department of Computer Science

Date: 13th April 2020

Midterm Assignment –Spring 2020

Course Title: Differential Equations

Instructor: Engr. Latif Jan

Program: BS (CS-SE-EE) **Total Marks: 30** **Time Allowed: 6 days Note:**

Attempt all Questions:

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Q 1: a) Define differential equation along with 2 examples? **(1+1 Marks)** **b)** Define a Separable Differential Equation (DE)? **(1+4+3 Marks)**

i. Solve the following **Initial Value Problem (IVP) using separable DE** and find the interval of validity of the solution.

$$\text{(a)}\ \ \boldsymbol{y' = \frac{xy^3}{\sqrt{1+x^2}}} \qquad \boldsymbol{y(0) = -1}$$

$$\text{(b)}\ \ \boldsymbol{y' = e^{-y}(2x-4)} \qquad \boldsymbol{(5) = 0}$$

Q 2: a) Solve the following IVP using Linear Differential method **(2+5+3 Marks)**

(i) **Explain the steps for solving Linear Differential Equation.** (ii) $\boldsymbol{co(x)y' + sin(x)y = 2cos^3(x)sin(x) - 1}$ $\boldsymbol{y\left[\frac{\pi}{4}\right] = 3\sqrt{2}}$, $\boldsymbol{0 \le x \le \frac{\pi}{2}}$

(iii) $\boldsymbol{x' + 2x = sint}$

Q 3: Solve the following IVP for the exact equation and find the interval of validity for the solution. **(5+5 Marks)**

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad (0) = -3$

(ii) $\frac{2ty}{t^2+1} - 2t - (2 - l(t^2+1))y' = 0 \quad y(5) = 0$

Q1 : a) Define differential equation along with 2 examples? (1)

* **Differential Equation :-**

Differential equation is an equation that relates one or more functions and their derivatives. In applications, the function generally represent physical quantities, the derivatives represents their rates of change and the differential equation defines a relationship between the two.

Example :-

(1) $$\frac{dy}{dx} + xy = e^{2x}$$

(2) $$U_{xx} + U_{yy} + 0$$

As both of the above, example contains the derivative so these are differential equation.

b) Define a Separable Differential Equation?

* **Separable (D.E) :-**

A separable differential equation is one that can be broken into a set of separate equations of lower dimensionality by a method of separation of variables.

(2)

1. Solve the following initial value problem (IVP) using separable DE and find the interval of validity of the solution.

a) $y' = \frac{xy^3}{\sqrt{1+x^2}}$ $y(0) = -1$

Sol

$$\int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}} dx$$

$$\Rightarrow \int y^{-3} dy = \int \frac{x}{\sqrt{1+x^2}} dx$$

$\Rightarrow 1+x^2 = u$

$2x\, dx = du$

$\Rightarrow x\, du = \frac{du}{2}$

$$\Rightarrow \int y^{-3} dy = \int \frac{1}{\sqrt{u}} \frac{du}{2}$$

$$\Rightarrow \frac{y^{-3+1}}{-3+1} = \frac{1}{2} \frac{u^{-1/2+1}}{-1/2+1} + C$$

$$\Rightarrow \frac{y^{-2}}{-2} = \frac{2}{2} \sqrt{u} + C$$

$$\Rightarrow \frac{1}{-2y^2} = \sqrt{u} + C$$

$$\Rightarrow \frac{1}{-2y^2} = \sqrt{1+x} + C$$

$\Rightarrow y(0) = -1$

$$\frac{1}{-2(-1)^2} = \sqrt{1} + C$$

$$\Rightarrow \frac{1}{-2} = 1 + C$$

$$\Rightarrow -1 - \frac{1}{2} = C$$

$$\Rightarrow C = \frac{-2-1}{2}$$

$$\Rightarrow C = \frac{-3}{2}$$

$$\Rightarrow \frac{y^{-2}}{2} = \sqrt{1+x^2} - 3/2 \quad \underline{\text{An}}$$

b) $y' = e^{-y}(2x-4) \quad y(5) = 0$

Sol

$\Rightarrow \dfrac{dy}{dx} = e^{-y}(2x-4)$

$\Rightarrow \int \dfrac{dy}{e^{-y}} = \int (2x-4)\,dx$

$\Rightarrow \int e^{y}\,dy = \dfrac{2x^2}{2} - 4x + C$

$\Rightarrow e^{y} = x^2 - 4x + C$

$\Rightarrow y = \ln(x^2 - 4x) + C$

$\Rightarrow y = \ln((5)^2 - 4(5)) + C$

$\Rightarrow 0 = \ln(25-20) + C$

$\Rightarrow 0 = \ln 5 + C$

$\Rightarrow C = -\ln 5$

$\Rightarrow y = \ln(x^2 - 4x) - \ln 5$

(1) Q2 a) Solve the following IVP using Linear Differential method (5)

i) Explain the steps for solving Linear Differential Equation.

Following are the steps for solving Linear Differential Equation

1. Substitute $y = uv$, and

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

into $\frac{dy}{dx} + P(x)y = Q(x)$

2. Factor the parts involving v.
3. Involving v term equal to zero (this gives a D.E in u and x which can be solved in the next step).
4. Solve using separation of variable to find u.
5. Substitute u back into the equation we got at step 2.
6. Solve that to find v.
7. Finally substitute u and v into $y = uv$ to get solution.

ii) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1 \quad y\left[\frac{\pi}{4}\right] = 3\sqrt{2} \quad 0 \le x \le \pi/2$

Sol

$\div$ing by $\cos(x)$

$$\Rightarrow y' + \frac{\sin(x)y}{\cos(x)} = \frac{2\cos^3(x)\sin(x) - 1}{\cos(x)}$$

$$\Rightarrow y' + \tan(x)y = \frac{2\cos^3(x).\sin(x) - 1}{\cos(x)} \quad -(1)$$

It has the form $y' + P(x)y = Q(x)$

where $P(x) = \tan x$ & $Q(x) = \frac{2\cos^3(x).\sin(x) - 1}{\cos(x)}$

Integrating factor $= e^{\int P(x)\,dx} = e^{\int \tan x\,dx} = e^{\ln(\sec x)}$ ⑥

$\Rightarrow$ Integrating factor $= \sec x$.

Now Multiplying (1) by I.f or $\sec(x)$.

$(1) \Rightarrow \sec x\, y' + \sec x \tan(x) y = \dfrac{2\cos^3(x)\sin(x) - 1}{\cos(x)}$

$\Rightarrow \dfrac{d}{dx}[y \sec x] = \dfrac{2\cos^3 x \sin x - 1}{\cos^2 x}$

$\int d[y \sec x] = \int \dfrac{2\cos^3(x)\sin(x) - 1}{\cos^2 x}\,dx$

By solving this Integration we get.

$y \sec x = \dfrac{-1}{\tan^2 x + 1} - \tan x + C$

$y = \cos x\left[-\dfrac{1}{\tan^2 x + 1} - \tan x + C\right]$

Now $y(\pi/4) = 3\sqrt{2}$

$\Rightarrow y(\pi/4) = \cos \pi/4 \left[\dfrac{-1}{\tan^2(\pi/4) + 1} - \tan(\pi/4) + C\right]$

$\Rightarrow 3\sqrt{2} = \dfrac{1}{\sqrt{2}}\left[-\dfrac{1}{2} - 1 + C\right]$

$\Rightarrow 3 \times 2 = -\dfrac{3}{2} + C$

$\Rightarrow \boxed{C = -4}$

$y = \cos x\left[-\dfrac{1}{\tan^2 x + 1} - \tan x - 4\right], 0 \le x \le \pi/2$

iii) $x' + 2x = \sin t$.

Sol: $x' + 2x = \sin t \quad -(i)$

This D.E has the form $x' + P(t)x = Q(t)$

where $P(t) = 2$ and $Q(t) = \sin t$.

Integrating Factor $= e^{\int P(t)\,dt}$.

$I.f = e^{\int 2\,dt} = e^{2t}$

Multiplying (1) by e^{2t} we get.

$e^{2t}x' + 2xe^{2t} = e^{2t}\sin t$.

$\Rightarrow \frac{d}{dt}\left[xe^{2t}\right] = e^{2t}\sin t$

$\Rightarrow \int d\left[xe^{2t}\right] = \int e^{2t}\sin t\,dt$

$\Rightarrow xe^{2t} = \int e^{2t}\sin t\,dt$

Solving by Integration by parts method.

$\Rightarrow xe^{2t} = \frac{(2\sin t - \cos t)e^{2t}}{5} + C$

$\Rightarrow x = e^{-2t}\left[C + \frac{2e^{2t}\sin t}{5} - \frac{e^{2t}\cos t}{5}\right]$

Q3 Solve the following IVP for the exact equation and find (8) the interval of validity of the solution.

i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0 \quad y(0) = -3$

Sol

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0 \quad -(i)$$

$$\Rightarrow (2y + x^2 + 1)\frac{dy}{dx} = -2xy + 9x^2$$

$$\Rightarrow (2y + x^2 + 1)\,dy = (9x^2 - 2xy)\,dx$$

$$\Rightarrow (9x^2 - 2xy)\,dx - (2y + x^2 + 1)\,dy = 0$$

$$\Rightarrow (9x^2 - 2xy)\,dx + (-2y - x^2 - 1)\,dy = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = -2x$$

$$\frac{\partial N}{\partial x} = -2x$$

So, Exact Equation. Solution exists

$$\int M\,dx + \int (\text{term of } N, (\text{free of } x))\,dy$$

y-axis

$$\int (9x^2 - 2xy)\,dx + \int (-2y - 1)\,dy = C$$

$$\Rightarrow \frac{9x^3}{3} - \frac{\cancel{2}x^2y}{\cancel{2}} - \frac{\cancel{2}y^2}{\cancel{2}} - y = C$$

(9)

$$\frac{9x^3}{3} - x^2y - y^2 - y = C$$

$$y(0) = -3$$

$$-(-3)^2 - (-3) = C$$

$$-9 + 3 = C$$

$$\boxed{C = -6}$$

ii) $\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0 \qquad y(5) = 0$ (10)

Sol

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))\frac{dy}{dx} = 0$$

$$\left(\frac{2ty}{t^2+1} - 2t\right)dx - (2 - \ln(t^2+1))dy = 0$$

$$M(t,y) = \frac{2ty}{t^2+1} - 2t$$

$$N(t,y) = \ln(t^2+1) - 2$$

$$\frac{\partial M}{\partial y} = \frac{2t}{t^2+1}$$

$$\frac{\partial N}{\partial t} = \frac{2t}{t^2+1}$$

So Exact Equation, Solution exists

$\int M\,dx + \int$ (term of N free of x) dy

y-axis

$$\int\left(\frac{2ty}{t^2+1} - 2t\right)dt + \int -2\,dy = C$$

$$y\ln(t^2+1) - t^2 - 2y\ C$$

$$y(5) = 0$$

$$-(5)^2 = C$$

$$\boxed{C = -25}$$