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Q 1(a) Find the polynomial of degree 3 or less that interpolates the point $(0,2),(1,1)$ $(2,0)$ and $(3,-1)$.

sol: The Lagrange form is as fallows

$$P(x) = \frac{2(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} + \frac{1(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$$

$$+ \frac{0(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} - \frac{1(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}$$

$$= -\tfrac{1}{3}(x^3-6x^2+11x-6) + \tfrac{1}{2}(x^3-5x^2+6x) - \tfrac{1}{6}(x^3-3x^2+2x)$$

$$\boxed{= -x+2}$$

There exists exactly one interpolation polynomial of degree 3 or less, but it may or may not be exactly degree 3.

Q1 B Find the interpolating polynomial for the data points (0, 1), (2, 2) and (3, 4) for the given below.

Sol:

$$P_2(x) = \frac{1(x-2)(x-3)}{(0-2)(0-3)} + \frac{2(x-0)(x-3)}{(2-0)(2-3)}$$

$$+ 2\frac{(x-0)(x-3)}{(2-0)(2-3)} + 4\frac{(x-0)(x-2)}{(3-0)(3-2)}.$$

$$= \frac{1}{6}(x^2-5x+6) + 2\left(-\frac{1}{2}\right)(x^2-3x) + 4\left(\frac{1}{3}\right)(x^2-2x)$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x + 1.$$

$P_2(0) = 1$, $P_2(2) = 2$ and $P_2(3) = 4$

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Q2 A Use two point forward difference formula with $h=0.1$ to approximate the derivate of $f(x)=1/x$ at $x=2$.

Sol: The two-point forward-difference formula

$$f'(x) \approx \frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{2.1}-\frac{1}{2}}{0.1}$$

$$\simeq -0.2381$$

The difference between this approximation and the correct derivative $f'(x) = -x^{-2}$ at $x=2$ is the error

$$-0.2381-(-0.2500) = 0.0119$$

Compare this to the error predicted by the formula, which is $h f''(c)/2$ for some c between 2 and 2.1 since $f''(x) = 2x^{-3}$ the error must be between

$$(0.1)2^{-3} \simeq 0.0125 \quad \text{and} \quad (0.1)(2.1)^{-3} \simeq 0.0108$$

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Q2 B Use newton's divided difference to find the interpolation polynomial passing through the point (0,1) (2,2) and (3, 4).

Sol: Applying the definitions of divided difference leads to the following table

0	1	1/2	
2	2		1/2
3	4	2	

This table is computed as follows: After writing down the x and y coordinates in separate colums, calculate the next colums left to right as divided difference ~~as in (3.3)~~

$$\frac{2-1}{2-0} = \frac{1}{2}$$

$$\frac{2-\frac{1}{2}}{3-0} = \frac{1}{2}$$

$$\frac{4-2}{3-2} = 2$$

After completing the divided difference triangle, the coefficients of the polynomial 1, 1/2, 1/2 can be read from the top edge of the table. The interpolating polynomial can be written as

$$P(x) = 1 + \frac{1}{2}(x-0) + \frac{1}{2}(x-0)(x-2)$$

or in nested form

$$p(x) = 1 + (x-0)\left(\frac{1}{2} + (x-2)\,\frac{1}{2}\right)$$

The base points for the nested form are $x_1 = 0$ and $x_2 = 2$. Alternatively, we could do more algebra and write the interpolating polynomial as

$$P(x) = 1 + \frac{1}{2}x + \frac{1}{2}x(x-2) = \frac{1}{2}x^2 - \frac{1}{2}x + 1$$

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Q3 A least square Problem.

$$\begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 15 \\ 9 \end{pmatrix}$$

Sol: The normal equation $A^T A x = A^T b$

are

$$\begin{pmatrix} 9 & 6 \\ 6 & 29 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 45 \\ 75 \end{pmatrix}$$

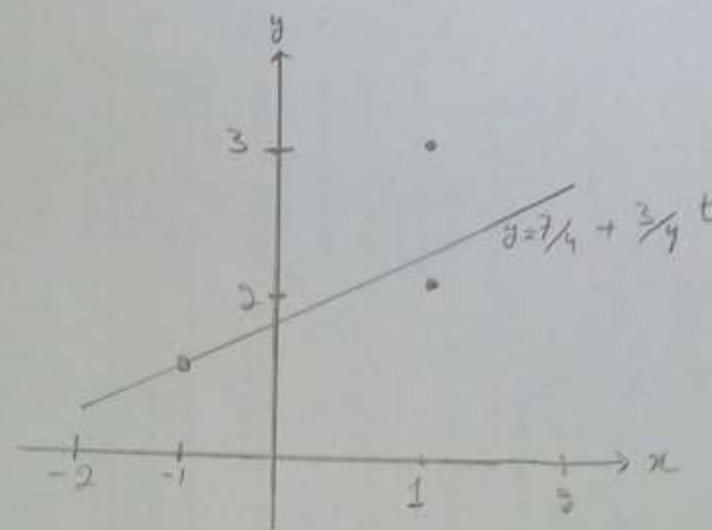
The solution of the normal equation are $\bar{x}_1 = 3.8$ and $\bar{x}_2 = 1.8$. The residual vector is

$$r = b - A\bar{x} = \begin{bmatrix} -3 \\ 15 \\ 9 \end{bmatrix} - \begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{pmatrix} \begin{bmatrix} 3.8 \\ 1.8 \end{bmatrix}$$

$$= \begin{pmatrix} -3 \\ 15 \\ 9 \end{pmatrix} - \begin{bmatrix} -3.4 \\ 13 \\ 11.2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 2 \\ -2.2 \end{bmatrix}$$

which has Euclidean norm $\|e\|_2 = \sqrt{(0.4)^2 + 2^2 + (-2.2)^2} = 3.$

Q3 B Find the line that best fits the three data points $(t, y) = (1, 2), (-1, 1)$ and $(1, 3)$ in the figure below.



Sol: The model is $y = c_1 + c_2 t$ and the goal is to find the best c_1 and c_2. Substitution of the data points into the model yields

$$c_1 + c_2(1) = 2$$
$$c_1 + c_2(-1) = 1$$
$$c_1 + c_2(1) = 3$$

or in matrix form

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$