NAME Muhammad iLyas

ID-No 15392

Degree B.S (Software Engr)

PAPER Differential Equation

QUESTION-No-1

PART 'A':-

2nd order Linear Homogeneous Equation:-

Consider a differential equation of type $y'' + py' + qy = 0$

where p, q are some constent coefficients. For each of the equation we can write the so-called characteristic equation:

$$k^2 + pk + q = 0$$

The general solution of homogenous differential equations depends on the roots of the characteristic quadriatic equation.

Example:-

Solve the differential equation

$$y'' - 6y' + 5y = 0$$

Sol:- First we write the corresponding characteristic equation of the given differential Equation

$$k^2 - 6k + 5 = 0$$

The Roots of this equation are $k_1 = 1$, $k_2 = 5$, Since the Roots are Real and distinct, the general solution has the form

$$y(x) = C_1 e^{x} + C_2 e^{5x}$$

where C_1 & C_2 are arbitrary constants.

2nd order linear Non-Homogeneous Equation:

A Linear non-homogenous second-order equation with variable coefficients has the form

$$y'' + a_1(x)y' + a_2(x)y = 0$$

wher $a_1(x)$, $a_2(x)$ and $f(x)$ are continous functions on the interval $[a, b]$

The associated homogeneous equation is written as

$$y'' + a_1(x)y' + a_2(x)y = 0$$

The general solution of the Non-homogeneous equation is the sum of the general solution $y_0(x)$ of the associated homogeneous equation and a particular solution $Y(x)$ of the non-homogenous equation

$$y(x) = y_0(x) + Y(x)$$

Example:-

Find the general solution of

$$x^2y'' - 2xy' + 2y = x^2 + 1 \text{ (for } x > 0)$$

Sol:-

$$x^2y'' - 2y' + 2y = 0$$

is the function $y_1 = x$ we find the second independent solution y_2 using the Liouville formula

$$W_{y_1,y_2}(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = C_1 \exp\left(-\int \frac{a_1(x)}{a_2(x)}\,dx\right)$$

Hence

$$y_2' y_1 - y_2 y_1' = C_1 e^{-\int\left(\frac{-2x}{x^2}\right)dx} = C_1 e^{2\ln|x|} = C_1 e^{\ln x^2}$$
$$= C_1 x^2$$

Divide both sides of y_1^2

$$\Rightarrow \frac{y_2' y_1 - y_2 y_1'}{y_1^2} = \frac{C_1 x^2}{y_1^2} = \frac{C_1 x^2}{x^2} = C_1 \Rightarrow \left(\frac{y_2}{y_1}\right)' = C_1$$

After integration we have

$$\Rightarrow \frac{y_2}{y_1} = C_1 x + C_2 \Rightarrow y_2 = y_1 (C_1 x + C_2) = x(C_1 x + C_2)$$
$$= C_1 x^2 + C_2 x$$

Thus the general solution of the homogeneous equation is given by the function

$$y_0(x) = C_1 x^2 + C_2 x,$$

where C_1, C_2 are arbitaray constants.

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Q(1) PART 'B'

(i)

$$4y'' - 6y' + 7y = 0$$

Sol:-

Lets find the root of the characteristic equation.

$$4\lambda^2 - 6\lambda + 7 = 0$$

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(4)(7)}}{2(4)}$$

$$\lambda = \frac{6 \pm \sqrt{36-112}}{8}$$

$$\lambda = \frac{6 \pm \sqrt{-76}}{8}$$

$$\lambda = \frac{6}{8} \pm \frac{\sqrt{19}\,i}{8}$$

$$\lambda = \frac{3}{4} \pm \frac{\sqrt{19}}{4}i$$

$$\lambda_1 = \frac{3}{4} + \frac{\sqrt{19}}{4}i, \quad \lambda_2 = \frac{3}{4} - \frac{\sqrt{19}}{4}i$$

So it is has the complex conjugate roots.

$$G_1(x) = e^{\lambda_1 x} \cos^{\lambda_1(x)}$$

$$G_2(x) = e^{\lambda_2 x} \sin^{\lambda_2(x)}$$

$$y = C_1 e^{\frac{3}{4}x} \cos^{i\frac{\sqrt{19}}{4}(x)} + e^{\frac{2x}{4}} \sin^{i\frac{\sqrt{19}}{4}(x)} C_2.$$

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Q1 ⇒ Part 'B' ⇒ (ii)

$$y'' - 4y' - 12y = 3e^{5x}$$

Solution:

The first thing that we're going to find the complementry solution to this differential Equation

$$y'' - 4y' - 12y = 0$$

The characteristic equation for this differenal Equation and its roots are,

$$r^2 - 4r - 12 = (r-6)(r+2) = 0$$

$$r_1 = -2, \quad r_2 = 6$$

The complementry solution is then

$$y_c(x) = C_1 e^{-2x} + C_2 e^{6x}$$

Since,

$$Y_p(x) = Ae^{(x)}$$

plug this into the differential equation and see if we can determine what A needs to be

$$\Rightarrow 25Ae^{5(x)} - 4(5Ae^{5(x)}) = 3e^{5x}$$

$$\Rightarrow -7Ae^{5x} = 3e^{5x}$$

$$\Rightarrow -7A = 3 \Rightarrow A = -\frac{3}{7}$$

A particular solution to the differential equation

$$Y_p(x) = -\frac{3}{7}e^{5x}$$

Ans

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QUESTION - No - 2

Solve the IVP for 2^{nd} order linear equation.

(i)

$$16y'' - 40y' + 25y = 0$$

$$y(0) = 3 \quad , \quad y'(0) = -\frac{9}{4}$$

Solution: The characteristic equation and its roots are,

$$\Rightarrow 16r^2 - 40r + 25 = (4r-5)^2 = 0 \quad r_{1,2} = \frac{5}{4}$$

$$= \quad y(t) = c_1 e^{\frac{5t}{4}} + C_2 t e^{\frac{5t}{4}}$$

$$= \quad y'(t) = \frac{5}{4} C_1 e^{\frac{5t}{4}} + C_2 e^{\frac{5t}{4}} + \frac{5}{4} C_2 t e^{\frac{5t}{4}}$$

Now,

$$3 = y(0) = c_1$$

$$-\frac{9}{4} = y'(0) = \frac{5}{4} c_1 + c_2$$

This system is easly solve to get $c_1 = 3$ and $c_2 = -6$. The actual solution to the IVP is

$$y(t) = 3e^{\frac{5t}{4}} - 6te^{\frac{5t}{4}}$$

Ans.

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(ii) $y'' + 14y' + 49y = 0$

$y(-4) = -1$, $y'(-4) = 5$

Solution:-

The characteristic equation and its roots are;

$= r^2 + 14r + 49 = (r+7)^2 = 0 \quad r_1, r_2 = 7$

The general solution and its derivative are

$= \quad y(t) = c_1 e^{-7t} + c_2 t e^{-7t}$

$= \quad y'(t) = -7c_1 e^{-7t} + c_2 e^{-7t} - 7c_2 t e^{-7t}$

Now plugging in the intial conditione.

$\Rightarrow \quad -1 = y(-4) = c_1 e^{28} - 4c_2 e^{28}$

$= 5 = y'(-4) = -7c_1 e^{28} + c_2 e^{28} + 28 c_2 e^{28} = -7c_1 e^{28}$
$+ \; 29 c_2 e^{28}$

Solving this system gives the following constants

$c_1 = -9e^{-28} \; , \; c_2 = -2e^{-28}$

The actual solution to the IVP is then

$= \quad y(t) = -9e^{-28} e^{-7t} - 2te^{-28} e^{-7t}$

$= \quad y(t) = -9e^{-7(t+4)} - 2te^{-7(t+4)}$

Ans.

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(iii) $y'' - 4y' + 9y = 0$

$y(0) = 0 \ , \ y'(0) = -8.$

Solution: The characteristic equation for this differential equation is

$$r^2 - 4r + 9 = 0$$

The roots of this equation are

$r_{1,2} = 2 \pm \sqrt{5}i$. So

$$= y(t) = C_1 e^{2t} \cos(\sqrt{5}t) + C_2 e^{2t} \sin(\sqrt{5}t)$$

So,

$$0 = y(0) = C_1$$

In other words, the first term will drop out in order to meet the first condition. This make the solution along with its derivate.

$$= y(t) = C_2 e^{2t} \sin(\sqrt{5}t)$$

$$= y'(t) = 2c_2 e^{2t} \sin\sqrt{5}t + \sqrt{5} c_2 e^{2t} \cos(\sqrt{5}t)$$

Now apply the second condition to the derivate to get.

$$-8 = y'(0) = \sqrt{5} c_2 \Rightarrow c_2 - \frac{8}{\sqrt{5}}$$

The actual solution is then

$$y(t) = -\frac{8}{\sqrt{5}} e^{2t} \sin(\sqrt{5}t)$$

Ans.

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(iv) $y'' - 8y' + 17y = 0$

$y(0) = -4, \; y'(0) = -1$

Solution:

The characteristics equation This time is,

$$r^2 - 8r + 17 = 0$$

The Roots of this are $r_{1,2} = 4 \pm i$.
The general solution as well as its derivate

$$= y(t) = c_1 e^{4t} \cos(t) + c_2 e^{4t} \sin(t)$$

$$= y'(t) = 4c_1 e^{4t} \cos(t) - c_1 e^{4t} \sin(t) + 4c_2 e^{4t} \sin(t) + c_2 e^{4t} \cos(t)$$

This time we will need the derivative from the start as we won't be having one of the terms drop out. Applying the initial conditions gives the following system.

$$-4 = y(0) = c_1$$
$$-1 = y'(0) = 4c_1 + c_2$$

Solving this system gives $c_1 = -4$ and $c_2 = 15$
The actual solution to the IVP is

$$y(t) = -4e^{4t} \cos(t) + 15e^{4t} \sin(t)$$

Ans.

QUESTION - NO - 3

Laplace transform:-

The method of laplace transforms is a system that relies on algebra (rather than calculus-based methods) to solve linear differential equations.

Definition:-

Let $f(t)$ be defined for $t \geq 0$. The laplace transform of $f(t)$ denoted by $F(s)$ or $\mathcal{L} f(t)$, is an integral transform given by the laplace integral.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The laplace transform is an operation that transform a function of t (i.e, a function of time domain) defined as $[0, \infty]$ to a function of s (i.e, of frequency domain). $F(s)$ is the laplace transform.

Example:-

Let $f(t) = 1$ then $F(s) = \frac{1}{s}$ $\quad s > 0$

Sol:-

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} dt$$

$$= \frac{-1}{s} e^{-st} \Big|_0^\infty$$

The integral is divergent whereever $s \leq 0$. However, when $s > 0$ it

$$\frac{-1}{s}(0 - e^0) = \frac{-1}{s}(-1) = \frac{1}{s} = F(s)$$

A - Find the Laplace transforms.

(i)

$f(t) = 6(e^{-5t}) + e^{3t} + 5(t^3) - 9$

Sol:-

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$F(s) = \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

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(ii)

$g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

Sol:

$$G(s) = 4\frac{s}{s^2+4^2} - 9\frac{4}{s^2+4^2} + 2\frac{s}{s^2+10^2}$$

$$G(s) = \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

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(iii)

$h(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t)$

Sol:

$$H(s) = \frac{1}{s-3} + \frac{s}{s^2+6^2} - \frac{s-3}{(s-3)^2+6^2}$$

$$H(s) = \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

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QUESTION - 4

Solving the following IVP using Laplace Transform.

(i)

$$y'' - 10y' + 9y = 5t$$

$$y(0) = -1 \quad , \quad y'(0) = 2$$

Solution:

Take the tranform of every term in differential equation.

$$= \mathcal{L}\{y''\} - 10\,\mathcal{L}\{y'\} + 9\,\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

Using the appropriate farmula from our table of laplace tranform.

$$= s^2Y(s) - sy(0) - y'(0) - 10(sY(s) - y(0)) + 9Y(s) = \frac{5}{s^2}$$

$$(s^2 - 10s + 9)\,Y(s) + s - 12 = \frac{5}{s^2}$$

Solve for $Y(s)$

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

Combining the two terms gives us.

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

partial fraction decompostion for this transform is.

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Setting numerators equal gives.

$$=$$

$$5 + 12s^2 - s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

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Picking appropriate values of s and solving for the constant gives

$S = 0 \qquad 5 = 9B \implies B = \frac{5}{9}$

$S = 1 \qquad 16 = -8D \implies D = -2$

$S = 9 \qquad 248 = 648C \implies C = \frac{31}{81}$

$S = 2 \qquad 45 = -14A + \frac{4345}{81} \implies A = \frac{50}{81}$

Plugging in the constants gives

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

The solution to the IVP

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^{t}$$

Ans.

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(ii) $y'' - 6y' + 15y = 2\sin(3t)$

$y(0) = -1 \quad , \quad y'(0) = -4$

Solution:-

Take laplace Transform of everything

$\Rightarrow s^2Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 15Y(s) = 2\frac{3}{s^2+9}$

$= (s^2 - 6s + 15)Y(s) + s - 2 = \frac{6}{s^2+9}$

Solve for $Y(s)$,

$= Y(s) = \dfrac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)}$

Now, do the partial fraction on this

$= Y(s) = \dfrac{As+B}{s^2+9} + \dfrac{Cs+D}{s^2-6s+15}$

Now, setting numerator equal gives

$\Rightarrow -s^3 + 2s^2 - 9s + 24 = (As+B)(s^2-6s+15) + (Cs+D)(s^2+9)$

$= (A+C)s^3 + (-6A+B+D)s^2 + (15A - 6B + 9C)s + 15B + 9D$

setting cofficients equal and solving

$s^3 : \quad A + C = 1$

$s^2 : \quad -6A + B + D = 2$

$s^1 : \quad 15A - 6B + 9C = -9$

$s^0 : \quad 15B + 9D = 24$

$\Rightarrow A = \frac{1}{10}$

$\Rightarrow B = \frac{1}{10}$

$\Rightarrow C = -\frac{11}{10}$

$\Rightarrow D = \frac{5}{2}$

Now, plug these into the decomposition.

$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s + 25}{s^2+6s+15}\right)$$

$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11(s-3+3)+25}{(s-3)^2+6}\right)$$

$$\Rightarrow \frac{1}{10}\left(\frac{s}{s^2+9} + \frac{1\,\frac{3}{3}}{s^2+9} - \frac{11(s-3)}{(s-3)^2+6} - \frac{8\frac{\sqrt{6}}{\sqrt{6}}}{(s-3)^2+6}\right)$$

Finally take the invese transform.

$$y(t) = \frac{1}{10}\left(\cos(3t) + \frac{1}{3}\sin(3t) - 11e^{3t}\cos(\sqrt{6}t) - \frac{8}{\sqrt{6}}e^{3t}\sin(\sqrt{6}t)\right)$$

Ans.

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