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SEC - B

CIVIL (4<sup>th</sup> SEM)

MAM SHUMAILA

DIFFERENTIAL EQUATION

30/06/2020



Q No: 1

①

## The wave Equation:

The wave equation is an important second order linear partial differential equation for the description of wave as they occur in

classical physics - such

as mechanical wave

e.g. water waves sound

wave & seismic wave.

⇒ We generally visit

beach & if we stand

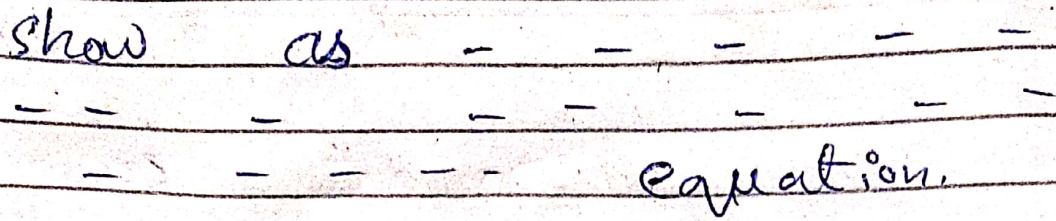
on an ocean shore

& take a snapshots



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the waves the picture

show as  equation.

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

i)  $w = \sin(x+ct) + \cos(2x+2ct)$

ii)  $w = \tan(2x+ct)$

i  $w = \sin(x+ct) + \cos(2x+2ct)$ .

Sol:-

$$w = \sin(x+ct) + \cos(2x+2ct)$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) + c - \sin(2x+2ct) + 2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) + c^2 - \cos(2x+2ct) + 4c^2$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct) + 2$$



$$\Rightarrow \frac{\partial^2 \omega}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

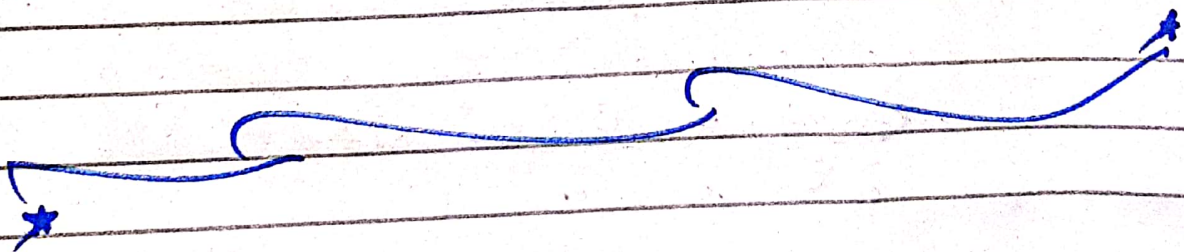
$$= \left[ -\sin(x+ct) - 4\cos(2x+2ct) \right]$$

$$\frac{\partial^2 \omega}{\partial t^2} = +c^2 \left[ -\sin(x+ct) - 4\cos(2x+2ct) \right]$$

$$c^2 \cdot \frac{\partial^2 \omega}{\partial x^2}$$

Hence

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$$





$$u:- \quad w = \tan(2x + ct).$$

Sol:-

$$w = \tan(2x + ct)$$

A we know that

$$w = \tan(2x + ct)$$

Taking derivative w.r.t  $x$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \tan(2x + ct) \cdot c$$

$$\Rightarrow \frac{\partial w}{\partial x} = \sec^2(2x + ct) \cdot c$$

$$\frac{\partial w}{\partial x} = c \sec^2(2x + ct)$$

Again Taking Derivative.

$$\frac{\partial^2 w}{\partial x^2} = c \frac{\partial}{\partial x} \sec^2(2x + ct)$$



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$$\frac{\partial^2 w}{\partial t^2} = c \cdot 2 \sec(2x+ct) - \sec(2x+ct) \times \tan(2x+ct) c$$

$$\frac{\partial^2 w}{\partial t^2} = 2c^2 \sec^2(2x+ct) - \tan(2x+ct)$$

Now

$$w = \tan(2x+ct)$$

Taking derivative w.r.t x

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \tan(2x+ct)$$

$$\frac{\partial w}{\partial x} = \sec^2(2x+ct) \cdot 2$$

$$\frac{\partial w}{\partial x} = 2 \sec^2(2x+ct)$$

⇒ Again taking derivative

$$\frac{\partial^2 w}{\partial x^2} = 2 \frac{\partial}{\partial x} \sec^2(2x+ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 2 \cdot 2 \sec(2x+ct) \sec(2x+ct) \tan(2x+ct) \cdot 2$$



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$$\Rightarrow \frac{\partial^2 \omega}{\partial t^2} = G \sec^2(2\pi ft) \tan(2\pi ft)$$

As we know the  
wave Equation

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$$

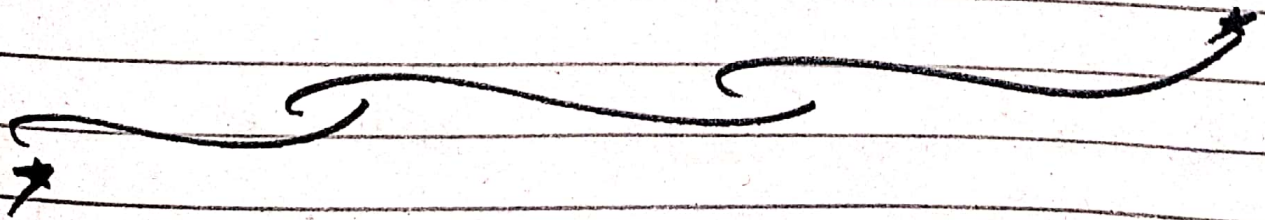
Put values in wave  
equation.

$$1 \neq 3$$

Hence

$$L.H.S \neq R.H.S.$$

not proof wave  
equation.





Q No: 2

Expand the following function in a fourier Series.

Soln:

$$f(x) = x, \quad -\pi < x < 0$$

$$= 2x, \quad 0 < x < \pi$$

$$a_0 = ? \quad a_n = ? \quad b_n = ?$$

Now

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 x dx$$

$$+ \frac{1}{\pi} \int_0^{\pi} 2x dx$$

$$= \frac{1}{\pi} \left[ \frac{x^2}{2} \right]_{-\pi}^0 + \frac{2}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[ 0 - \frac{\pi^2}{2} \right] + \frac{2}{\pi} \left[ \frac{\pi^2}{2} - 0 \right]$$

$$a_0 = -\frac{\pi}{2} + \pi = \frac{\pi}{2} \rightarrow \textcircled{1}$$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= -\frac{1}{\pi} \left[ x \int_{-\pi}^0 x \cos nx \, dx \right] + \frac{1}{\pi} \int_0^{\pi} (2x \cos nx) \, dx$$

$$= -\frac{1}{\pi} \left[ x \left( \frac{\sin nx}{n} \right) - \left( -\frac{\cos nx}{n^2} \right) \right]_{-\pi}^0 +$$

$$\frac{2}{\pi} \left[ x \left( \frac{\sin nx}{n} \right) - \left( -\frac{\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$a_n = \frac{1}{\pi} \left[ \frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2} + \frac{2}{\pi} \left[ \frac{\cos n\pi - \cos(0)}{n^2} \right] \right]$$

$$= \frac{1}{\pi} \left[ \frac{1 - (-1)^n}{n^2} + \frac{2(-1)^n - 2}{n^2} \right]$$

$$= \frac{(-1)^n - 1}{\pi n^2}$$

So

$$a_n = \begin{cases} \frac{-2}{\pi n^2} & ; \text{ if } n \text{ is odd} \\ 0 & ; \text{ } n \text{ is even} \end{cases}$$



$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx$$

$$+ \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi} \left[ x \left( \frac{-\cos nx}{n} \right) - \left( \frac{-\sin nx}{n^2} \right) \right]_{-\pi}^{\pi}$$

$$+ \frac{2}{\pi} \left[ x \left( \frac{-\cos nx}{n} \right) - \left( \frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$b_n = \frac{1}{\pi} \left[ \frac{-\pi \cos n\pi}{n} \right] + \frac{2}{\pi} \left[ \frac{-\pi \cos n\pi}{n} \right]$$

$$= \frac{-3 \cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}$$

Hence.

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos (2n-1)x + 3}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Required  
Fourier Series



Q No: 3

Solve the initial  
value problem

$$y'' - 4y' + 13y = 8\sin 3x$$

$$y(0) = 1, \quad y'(0) = 2$$

Sol:-

$$y'' - 4y' + 13y = 8\sin 3x \rightarrow (1)$$

Associated Homogenous Eq (1) is

$$y'' - 4y' + 13y = 0 \rightarrow (2)$$

Into Auxiliary Equation

$$\text{put } y = m \text{ in } \rightarrow (2)$$

$$m^2 - 4m + 13 = 0$$

$\Rightarrow$  Use Quadratic Formula.

$$a = 1, \quad b = -4, \quad c = 13.$$



$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2}$$

$$m = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$m = \frac{4 \pm \sqrt{-36}}{2}$$

$$m = \frac{4 \pm \sqrt{36}i}{2} = \frac{4 \pm 6i}{2}$$

$$m = 2 \pm 3i$$

$$m_1 = 2 + 3i, m_2 = 2 - 3i$$

$$y_e = e^{2x} (C_1 \cos 3x + C_2 \sin 3x) \rightarrow A$$

$$\text{Let } y_p = A \cos 3x + B \sin 3x \rightarrow A$$

Differentiate w.r.t "x"

$$y_p = -3A \sin 3x + 3B \cos 3x.$$



Again Differentiate w.r.t  $x$ .

$$y_p'' = -9A \cos 3x + 3B \cos 3x$$

put in Eq (1)

$$(-9A \cos 3x - 9B \sin 3x) - 4(-3A \sin 3x + 3B \cos 3x) + 13(C \cos 3x + B \sin 3x) - 8 \sin 3x$$

$$(-9A - 12B + 13A) \cos 3x + (-9B + 12A + 13B) \sin 3x = 8 \sin 3x$$

$$(4A - 12B) \cos 3x + (4B + 12A) \sin 3x = 8 \sin 3x$$

Comparing Coefficient.

$$\sin 3x \Rightarrow 4B + 12A = 8 \rightarrow a$$

$$\cos 3x \Rightarrow 4A - 12B = 0$$

$$4A = 12B \Rightarrow \boxed{A = 3B} \rightarrow *$$

put in \* Eq a

$$4B + 12(3B) = 8$$

$$4B + 36B = 8$$

$$40B \Rightarrow B = \frac{1}{5} \rightarrow c$$

Put Equation c in eq (\*).



$$A = \frac{3}{5} \rightarrow (d)$$

Put  $c$  &  $d$  in (\*)

$$y_p = \frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \rightarrow (B)$$

The General Solution is

$$y = y_c + y_p.$$

$$y = e^{2x} (C_1 \cos 3x + C_2 \sin 3x) + \frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x$$

Now we have to find the values of  $C_1$  &  $C_2$

for this.

Put  $x=0$  &  $y=1$  in (c)

$$1 = e^{n(0)} (C_1 \cos 3(0) + C_2 \sin 3(0)) + \frac{3}{5} \cos 3(0) + \frac{1}{5} \sin 3(0)$$



$$I = (C_1 C_1)' + C_2 (0) + \frac{3 C_1}{5} + \frac{1 C_2}{5}$$

$$I = C_1 + \frac{3}{5}$$

$$C_1 = \frac{2}{5} \rightarrow (*)$$

Differentiate C w.r.t x

$$y' = C_1 (2e^{2x} \cos 3x - 3e^{2x} \sin 3x) + C_2 (2e^{2x} \sin 3x + 3e^{2x} \cos 3x) - \frac{6}{5} \sin 3x + \frac{3}{5} \cos 3x \rightarrow (D)$$

Put  $y' = 2$ ,  $x = 0$  in (D)

$$y' = C_1 (2e^{2x} \cos 3x - 3e^{2x} \sin 3x) + C_2 (2e^{2x} \sin 3x + 3e^{2x} \cos 3x) - \frac{6}{5} \sin 3x + \frac{3}{5} \cos 3x$$

Put  $y' = 2$ ,  $x = 0$

$$2 = C_1 (2e^{2(0)} \cos 3(0) - 3e^{2(0)} \sin 3(0)) - \frac{6}{5} \sin 3(0) + \frac{3}{5} \cos 3(0)$$



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$$2 = C_1(2) + C_2(3) - 0 + \frac{3}{5}$$

$$2 = 2C_1 + 3C_2 + \frac{3}{5}$$

Put  $C_1 = \frac{2}{5}$

$$2 = \frac{4}{5} + 3C_2 + \frac{3}{5}$$

$$2 = \frac{7}{5} + 3C_2$$

$$3C_2 = \frac{2-7}{5} \Rightarrow C_2 = \frac{3}{15} \rightarrow \frac{1}{5}$$



Put  $*1$  &  $*2$  in c.

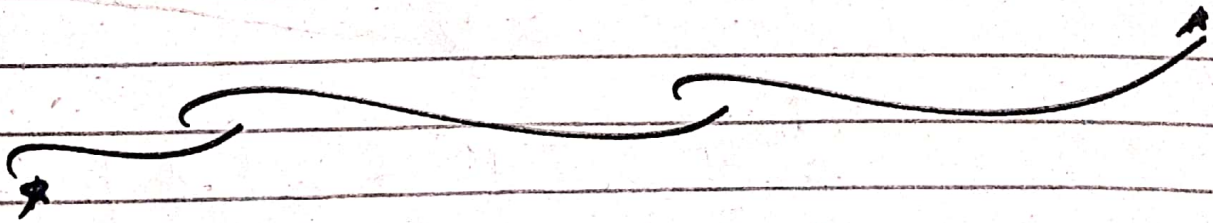
$$y = e^{2x} \left( \frac{2}{5} \cos 3x + \frac{3}{15} \sin 3x \right) + \frac{3}{5} \cos 3x$$

$$+ \frac{1}{5} \sin 3x.$$

$$y = \frac{2}{5} e^{2x} \cos 3x + \frac{3}{15} e^{2x} \sin 3x + \frac{3}{5} \cos 3x$$

$$+ \frac{1}{5} \sin 3x.$$

Required General Solution.





$$(D^2 - DD')Z = \cos x \cos 2y.$$

Sol:-

$$(D^2 - DD')Z = \cos x \cos 2y.$$

In CE is given by.

$$CF = \phi_1(y) + \phi_2(y+x)$$

while its PI is given by.

$$PI = \frac{1}{(D^2 - DD')} \cdot \frac{1}{2} \left[ \cos(x+2y) + \cos(x-2y) \right]$$

$$= \frac{1}{2} \left[ \frac{1}{(-1+2)} \cos(x+2y) + \frac{1}{(-1-2)} \right]$$

$$\cos(x-2y) \Big]$$

$$= \frac{1}{2} \cos(x+2y) - \frac{1}{6} \cos(x-2y)$$

Complete solution of the given PDE'S



$$z = \phi_1(y) + \phi_2(y+k) + \frac{1}{2}$$

$$\cos(x+2y) - \frac{1}{6} \cos$$

$$(x-2y) \text{ Ans.}$$

