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Subjeet Differential equation

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Summer Mid term

Q1) Solve the initial value program

$$\frac{dy}{dt} = e^{y-t} \sec(y)(1+t^2) \quad y(0) = 0$$

Solution:-

$$dy = e^{y} \cdot e^{-t} \sec(y)(1+t^2)\,dt$$

$$\frac{1}{e^{y} \cdot \sec(y)}\,dy = (1+t^2)e^{-t}\,dt$$

As $\cos(y) = \frac{1}{\sec(y)}$

$$\int e^{-y} \cos y\,dy = \int (1+t^2)\,e^{-t}\,dt$$

using integration by parts

$$e^{-y} \int \cos y\,dy - \int \left(\int \cos y \cdot \frac{d}{dy} e^{-y}\right) = (1+t^2) \int e^{-t}$$

$$- \int \left(\int e^{-t} \cdot \frac{d}{dt}(1+t^2)\right) \quad \text{— eq (1)}$$

$$e^{-y} \int \cos y\,dy - \int \left(\int \cos y \cdot \frac{d}{dy} e^{-y}\right)$$

$$e^{-y} \sin y - \int (\sin y \cdot e^{-y}(-1))$$

$$e^{-y} \sin y + \int (\sin y \cdot e^{-y})$$

$$e^{-y} \sin y + \int (e^{-y} \sin y)$$

Again integration by parts

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$$e^{-y}\sin y + e^{-y}(-\cos y) - \int\left(\int \sin y \frac{d}{dy} e^{-y}\right)$$

$$e^{-y}\sin y + e^{-y}(-\cos y) - \int\left(-\cos y \frac{e^{-y}}{-1}\right)$$

$$e^{-y}\sin y - e^{-y}\cos y - \int(\cos y\, e^{-y})$$

since $\int(\cos y\, e^{-y}) = L.H.S$

LHS will become

$$LHS = e^{-y}(\sin y - \cos y) - LHS$$

$$2LHS = e^{-y}(\sin y - \cos y)$$

$$LHS = \frac{e^{-y}(\sin y - \cos y)}{2}$$

Now taking RHS

$$\int(1+t^2)e^{-t}dt$$

$$(1+t^2)\int e^{-t} - \int\left(\int e^{-t}\cdot\frac{d}{dt}(1+t^2)\right)$$

$$-(1+t^2)e^{-t} - \int(-e^{-t}(2t))$$

$$-(1+t^2)e^{-t} + \int((2t)e^{-t})$$

using integration by parts

$$-(1+t^2)e^{-t} + \left(2t\int e^{-t} - \int\left(\int e^{-t}\frac{d}{dt}2t\right)\right)$$

$$-(1+t^2)e^{-t} + \left(-2t\,e^{-t} - \int(-e^{-t}2\right)$$

$$-(1+t^2)e^{-t} + \left(-2te^{-t} + \int(2e^{-t})\right)$$

$$-(1+t^2)e^{-t} + \left(-2t\,e^{-t} - 2e^{-t}\right) + c$$

$$-(1+t^2)\,e^{-t} - 2t\,e^{-t} - 2e^{-t} + c$$

$$-e^{-t} - e^{-t}t^2 - 2te^{-t} - 2e^{-t} + c$$

$$-(t^2+2t+3)e^{-t} + c = RHS$$

Take LHS = RHS

$$\frac{e^{-y}(\sin y - \cos y)}{2} = -(t^2+2t+3)e^{-t} + c$$

As we know that

$$\Rightarrow \frac{1}{2}(-1) = -3+c$$

$$-\frac{1}{2} = -3+c$$

$$-\frac{1}{2} + 3 = c$$

$$c = \frac{5}{2}$$

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Therefore

$$\frac{e^{-y}}{2}\left(\sin(y) - \cos(y)\right) = -e^{-t}\left(t^2+2t+3\right) + \frac{5}{2}$$

(5)

Q2 Solve

$$(\sqrt{x+y}+\sqrt{x-y})dx - (\sqrt{x+y} - \sqrt{x-y}\, dy = 0$$

Solution :-

$$\frac{dy}{dx} = \frac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x+y} - \sqrt{x-y}}$$

This is homogeneous Differential eq in x and y to solve this put

$$y = Vx$$

$$\Rightarrow \frac{dy}{dx} = V + x\frac{dv}{dx}$$

Thus eq (1) becomes

$$V + x\frac{dv}{dx} = \frac{\sqrt{x+Vx} + \sqrt{x-Vx}}{\sqrt{x+Vx} - \sqrt{x-Vx}}$$

$$V + x\frac{dV}{dx} = \frac{\sqrt{1+V} + \sqrt{1-V}}{\sqrt{1+V} - \sqrt{1-V}}$$

$$V + x\frac{dv}{dx} = \frac{\sqrt{1+V} + \sqrt{1-V}}{\sqrt{1+V} - \sqrt{1-V}} \times \frac{\sqrt{1+V} + \sqrt{1-V}}{\sqrt{1+V} + \sqrt{1-V}}$$

$$V + x\frac{dv}{dx} = \frac{1 + \cancel{V} + 1 - \cancel{V} + 2\sqrt{1-V^2}}{2V}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x}\left(1+\sqrt{1-v^2}\right)}{\cancel{x}v}$$

$$v + x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}}{v}$$

$$x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}}{v} - v$$

$$x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)}{v}$$

$$\frac{v\,dv}{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)} = \frac{dx}{x}$$

Taking integral on both sides

$$\int \frac{v\,dv}{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)} = \int \frac{dx}{x}$$

Put $1+\sqrt{1-v^2} = t$

$$\Rightarrow \frac{1}{2}(1-v^2)^{-1/2}(-2v)\,dv = dt$$

$$\frac{v\,dv}{\sqrt{1-v^2}} = -dt$$

$$\int -\frac{dt}{t} = \int \frac{dx}{x}$$

$$-\ln t = \ln x + \ln c$$

$$-\ln\left(1+\sqrt{1-v^2}\right) = \ln Cx$$

$$\ln\left(1+\sqrt{1-v^2}\right) = -\ln cx$$

$$\cancel{\ln}\left(1+\sqrt{1-v^2}\right) = \cancel{\ln}(cx)^{-1}$$

$$1+\sqrt{1-v^2} = \frac{1}{cx}$$

$$1+\sqrt{1-\frac{y^2}{x^2}} = \frac{1}{cx}$$

$$1+\frac{\sqrt{x^2-y^2}}{x^2} = \frac{1}{cx}$$

$$x+\sqrt{x^2-y^2} = \frac{1}{c}$$

$$x+\sqrt{x^2-y^2} = c_1 \quad \because \frac{1}{c} = c_1$$

Which is a required Solution.

Q3 Solve

$$(D^4 + D^2)y = 3x^2 + 4\sin x - 2\cos x$$

Solution:-

$$(D^4 + D^2)y = 3x^2 + 4\sin x - 2\cos x$$

$$\Rightarrow f(D)y = f(x)$$

As it is non-homogeneous linear equation

Solution will be

$$y = y_c + y_p \quad \text{—(i)}$$

Complementary solution y_c

$$D^4 - D^2 = 0 \Rightarrow D^2(D^2+1) = 0$$

$$D^2 = 0 \Rightarrow \boxed{D = 0}$$

$$D^2 + 1 = 0 \Rightarrow D^2 = -1$$

$$D = \sqrt{-1} \Rightarrow \boxed{D = i} \text{ or } \boxed{D = 0 + i}$$

Roots are real and complex

$$y_c = c_1 e^{0x} + e^{0x}(c_2 \cos x + c_3 \sin x)$$

$$y_c = c_1 + c_2 \cos x + c_3 \sin x$$

$$y_p = \frac{1}{f(D)} F(x)$$

$$y_p = \frac{1}{D^4+D^2}(3x^2 + 4\sin x - 2\cos x)$$

$$= \frac{3x^2}{D^4+D^2} + \frac{4\sin x}{D^4+D^2} - \frac{2\cos x}{D^4+D^2}$$

$$f(D) = D^4 + D^2$$

at $D = 0 \Rightarrow f(D) = 0$

So $f'(D) = 4D^3 + 2D$

Now for $D = 0 \Rightarrow f'(D) = 0$

Again Differentiating

$$f''(D) = 12D + 2$$

so for $D = 0$

$$f''(0) = 12(0) + 2 = 2$$

so replacing $\frac{1}{f(D)}$ with $\frac{x^2}{f''(D)}$

$$\Rightarrow y_p = \frac{x^2\, 3x^2}{12D+2} + \frac{x^2}{12D+2} \cdot 4\sin x - \frac{x^2}{12D+2} \cdot 2\cos x$$

Putting $D = 0$ in all

So

$$y_p = \frac{x^2 \cdot 3x^2}{12(0)+2} + \frac{x^2 \cdot 4\sin x}{12(0)+2} - \frac{2x^2\cos x}{12(0)+2}$$

$$y_p = \frac{3x^4}{2} + \frac{4x^2\sin x}{2} - \frac{2x^2\cos x}{2}$$

$$= \frac{3x^4}{2} + \frac{4x^2\sin x}{2} - \frac{2x^2\cos x}{2}$$

$$= \frac{3}{2}x^4 + 2x^2\sin x - x^2\cos x$$

Put in eq (1)

$$y = C_1 + C_2\cos x + C_3\cos x + \frac{3}{2}x^4 + 2x^2\sin x - x^2\cos x$$

$$y = C_1 + (C_2 + x^2)\cos x + (C_3 + 2x^2)\sin x + \frac{3}{2}x^4$$

Ans