Q1: 1) $\frac{\partial^2 z}{\partial x \partial y}$ $\qquad z = \sin^{-1}\left(\frac{x}{y}\right)$

Sol:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{y\sqrt{y^2-x^2}}\right)$$

$$= \frac{-1}{y}\frac{\partial}{\partial x}\left[\frac{+x}{\sqrt{y^2-x^2}}\right]$$

$$= \frac{-1}{y}\left[\frac{1\cdot\left(\sqrt{y^2-x^2}\right) - x\cdot\frac{1}{2}\left(y^2-x^2\right)^{-1/2}\cdot -2x}{(y^2-x^2)}\right]$$

$$= \frac{\frac{-1}{y}\left(\sqrt{y^2-x^2} + \frac{x^2}{\sqrt{y^2-x^2}}\right)}{y^2-x^2}$$

$$= \frac{-1}{y}\left[\frac{y^2-x^2+x^2}{(y^2-x^2)(y^2-x^2)^{1/2}}\right]$$

$$= \frac{-y}{(y^2-x^2)^{3/2}}$$

(ii) $\frac{\partial^2 z}{\partial y \partial x}$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\sin^{-1}\frac{x}{y}\right)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y^2-x^2}}\right)$$

$$= \frac{0\cdot\sqrt{y^2-x^2} - 1\cdot\frac{1}{2}(y^2-x^2)^{-1/2}\cdot 2y}{\left(\sqrt{y^2-x^2}\right)^2}$$

$$= \frac{-y}{(y^2-x^2)(y^2-x^2)^{1/2}}$$

$$= \frac{-y}{(y^2-x^2)^{3/2}}$$

Q.2 $f(x,y) = e^x \sin y + e^y \cos x$

Sol: Laplace Equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ —— (a)

$z = e^x \sin y + e^y \cos x$

$$\frac{\partial z}{\partial x} = e^x \sin y + e^y(-\sin x)$$
$$= e^x \sin y - e^y \sin x$$
$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y - e^y \cos x$$

$$\frac{\partial z}{\partial y} = e^x \cos y + e^y \cos x$$
$$\frac{\partial^2 z}{\partial y^2} = e^x(-\sin y) + e^y \cos x$$
$$= -e^x \sin y + e^y \cos x$$

Now putting values in (a)

$$\left(e^x \sin y - e^y \cos x\right) + \left(-e^x \sin y + e^y \cos x\right) = 0$$
$$e^x \cancel{\sin y} - e^y \cancel{\cos x} - e^x \cancel{\sin y} + e^y \cancel{\cos x} = 0$$

Hence satisfied.

Q3: $f(x,y) = x^3 e^{-y} + y\sec x$

Sol:

$$\frac{\partial f}{\partial x} = 3x^2 e^{-y} + y\sec x\tan x$$

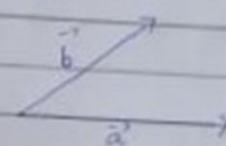
$$\frac{\partial f}{\partial y} = x^3(-1)e^{-y} + \sec x$$

$$= -x^3 e^{-y} + \sec x$$

Q4: $b = 6i + 3j + 2k$ onto $a = i - 2j - 2k$

Sol:

$$\text{Proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a}$$



$$= \frac{(6\hat{i} + 3\hat{j} + 2\hat{k})\cdot(\hat{i} - 2\hat{j} - 2\hat{k})}{\left(\sqrt{1^2 + (-2)^2 + (-2)^2}\right)^2}\cdot(\hat{i} - 2\hat{j} - 2\hat{k})$$

$$= \frac{6 - 6 - 4}{(1+4+4)}\cdot(\hat{i} - 2\hat{j} - 2\hat{k})$$

$$= \frac{-4}{9}(\hat{i} - 2\hat{j} - 2\hat{k})$$

Q5: $f(x,y) = xe^{y} + \cos(xy)$ at point $(2,0)$

$\vec{a} = 3i - 4j$

Sol:

The partial derivatives of f at point $(2,0)$ is

$$\frac{\partial f}{\partial x}(x,y) = e^{y} + (-\sin(xy).y)$$
$$= e^{y} - y\sin(xy)$$
$$\frac{\partial f}{\partial x}(2,0) = e^{0} - 0\sin(2.0)$$
$$= 1$$

$$\frac{\partial f}{\partial y}(x,y) = xe^{y} + (-\sin(xy).x)$$
$$= xe^{y} - x\sin xy$$
$$\frac{\partial f}{\partial y}(2,0) = 2e^{0} - 2\sin(2.0)$$
$$= 2 - 2.0$$
$$= 2$$

Therefore gradient is

$$\nabla f(2,0) = 1\hat{i} + 2\hat{j} = (1,2) \quad \text{—— (i)}$$

The directional derivative at $(2,0)$ in direction of a is

$$D_u f(2,0) = \nabla f(2,0).a \quad \text{—— (ii)}$$

$$a = \frac{3\hat{i} - 4\hat{j}}{\sqrt{3^2 + (-4)^2}} = \frac{3i - 4j}{\sqrt{9+16}} = \frac{3i-4j}{\sqrt{25}}$$

$$= \frac{3i-4j}{5}$$

Putting values in (ii)

$$D_u f(2,0) = (\hat{i} + 2\hat{j})\frac{(3i-4j)}{5} = \frac{3-8}{5} = \frac{-5}{5}$$
$$= -1$$

Q6: $f(x, y, z) = x^2 + y^2 + z^2 - 14$ and point $(1, -2, 3)$

Sol: $f = x^2 + y^2 + z^2 - 14$

$\vec{n} = \nabla f(1, -2, 3) = (f_x, f_y, f_z)$

$f_x = 2x$ and $f_x = 2(1) = 2$

$f_y = 2y$ " $f_y = 2(-2) = -4$

$f_z = 2z$ " $f_z = 2(3) = 6$

So required equation of tangent plane is:

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$2(x-1) + (-4)(y-(-2)) + 6(z-3) = 0$

$2(x-1) - 4(y+2) + 6(z-3) = 0$

$2x - 2 - 4y - 8 + 6z - 18 = 0$

$2x - 4y + 6z - 28 = 0$

$2x - 4y + 6z = 28$

Q7: $\int_0^1\int_0^1 xy+y^2\,dx\,dy$

Sol:

$$= \int_0^1 \left[\frac{x^2}{2}y \Big|_0^1 \right] \cancel{dy} + y^2x\Big|_0^1 \Big] dy$$

$$= \int_0^1 \left[\frac{1^2\cdot y^2}{2} - \cancel{\frac{0^2\cdot y^2}{2}} \right] \cancel{dy} + \left[y^2(1) - \cancel{y^2(0)} \right] \Big] dy$$

$$= \int_0^1 \frac{y^2}{2}+y^2\,dy$$

$$= \frac{y^3}{2\cdot 3}\Big|_0^1 + \frac{y^3}{3}\Big|_0^1$$

$$= \frac{y^3}{6}\Big|_0^1 + \frac{y^3}{3}\Big|_0^1$$

$$= \left[\frac{1^3}{6} - \cancel{\frac{0^3}{6}}\right] + \left[\frac{1}{3} - \cancel{\frac{0}{3}}\right]$$

$$= \frac{1}{6}+\frac{1}{3} = \frac{1+2}{6} = \frac{3}{6} = \frac{1}{2}$$

$3 \,|\, 3-6$
$\;\;\, 1-2$