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Q#1:-

Part A

① $W = \sin(x+ct) + \cos(2x+2ct)$

Given $\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2} \rightarrow ①$

Now

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{\partial}{\partial t}\left(\sin(x+ct)\right) + \frac{\partial}{\partial t}\left(\cos(2x+2ct)\right)$$

$$\frac{\partial w}{\partial t} = \left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$$

Now $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$

$$\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct)$$

Now

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\overset{c^2}{\cos}(2x+2ct)$$

1 $\Rightarrow$ $$-c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct)$$

$$= -c^2\left[-\sin(x+ct) - 4\cos(2x+2ct)\right]$$

$$-c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct) = -c^2 \sin(x+ct) - 4\cos(2x+2ct)$$

$\boxed{0 = 0}$ $\boxed{\text{satisfied}}$

Q# 01:-

Part B :- $W = \tan(2x + ct)$

Now $\frac{\partial w}{\partial t} = c \sec^2(2x + ct)$

and $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c \sec^2(2x + ct)\right)$

$= c_2^2 \cdot 2 \sec^2(2x + ct) \tan(2x + ct)$

Now $\frac{\partial w}{\partial x} = 2 \sec^2(2x + ct)$

$\frac{\partial^2 w}{\partial x^2} = 4 \sec^2(2x + ct) \tan(2x + ct)$

① $\Rightarrow$ $4c^2 \cancel{\sec^2}(2x + ct) \cancel{\tan}(2x + ct) =$

$4c^2 \cancel{\sec^2}(2x + ct) \cancel{\tan}(2x + ct)$

$\boxed{0 = 0}$ $\boxed{\text{Satisfied}}$

Q# 2:-

Expand the following function in a Fourier Series

$$F(x) = \begin{cases} x; & -\pi < x \leq 0 \\ 2x; & 0 \leq x \leq \pi \end{cases}$$

Sol:-

Fourier Series is written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left(\int_{\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx \right)$$

$$= \frac{1}{\pi} \left[\int_{\pi}^{0} x\, dx + \int_{0}^{\pi} 2x\, dx \right]$$

$$= 1/\pi \left(\frac{x^2}{2} \Big|_{-\pi}^{0} + x^2 \Big|_{0}^{\pi} \right)$$

$$= 1/\pi \left(-\frac{\pi^2}{2} + \pi^2 \right)$$

$$a_0 = 1/\pi \left(\pi^2/2 \right) = \pi/2$$

$$a_n = 1/\pi \int_{-\pi}^{0} f(x) \cos nx\, dx$$

$$= 1/\pi \int_{-\pi}^{0} x \cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} 2x \cos nx\, dx$$

$$= 1/\pi \left[x \int_{-\pi}^{0} \cos nx\, dx - \int_{0} \cos nx \frac{d}{dx}(2x) \right] dx$$

$$= 1/\pi \left[x \cancel{\frac{(\sin nx)}{n}}^{0} \Big|_{-\pi}^{0} - \int_{-\pi}^{0} \cos nx\, dx \right]$$

$$+ 1/\pi \left(\cancel{2x \cdot \frac{\sin nx}{n}}^{0} \Big|_{0}^{\pi} - \int \cos nx\, 2\, dx \right.$$

$$= -1/\pi \left[\cancel{\frac{\sin nx}{n}}^{0} \Big|_{-\pi}^{0} \right] + \frac{1}{\pi} \left[\cancel{\frac{\sin nx}{n}}^{0} \Big|_{0}^{\pi} \right.$$

$$\boxed{a_n = 0}$$

Now

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= 1/\pi \left[\int_{-\pi}^{0} x \sin n\, dx + \int_{0}^{\pi} 2x \sin n\, dx \right]$$

$$= \frac{1}{\pi}\left[\left(\cancel{\frac{\sin nx}{n}}^{0}\Big|_{-\pi}^{0} - x\frac{\cos(nx)}{n}\Big|_{-\pi}^{0}\right)\right]$$

$$+ \frac{1}{\pi}\left[2\cancel{\frac{\sin(nx)}{n}}^{0}\Big|_{0}^{\pi} - 2x\frac{\cos(nx)}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-x\frac{\cos(nx)}{n}\Big|_{-\pi}^{0}\right] + \frac{1}{\pi}\left[2x\frac{\cos(nx)}{n}\Big|_{0}^{\pi}\right]$$

$$= 1/\pi\left[-0\frac{\cos(n(0))}{n} - \left(-\pi\frac{\cos n(\pi)}{n}\right)\right]$$

$$+ 1/\pi\left[2\pi\frac{\cos(n\pi)}{n} - 2(0)\cancel{\frac{\cos(0)\pi}{n}}^{0}\right]$$

$$= 1/\pi\left[\pi\frac{\cos(n\pi)}{n}\right] + 1/\pi\left[2\pi\frac{\cos(n\pi)}{n}\right]$$

$$= 1/\pi\left[\pi\frac{\cos(n\pi)}{n} + 2\pi\frac{\cos(n\pi)}{n}\right]$$

$$= \frac{1}{\pi}\left[\pi + 2\pi\right]\frac{\cos(n\pi)}{n}$$

$$b_n = \frac{1}{\cancel{\pi}} \cdot 3\cancel{\pi}\frac{\cos(n\pi)}{n} = \frac{3(-1)^n}{n}$$

$$a_0 = \pi/2$$

$$a_n = 0$$

$$b_n = \frac{3(-1)^n}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cancel{\cos nx} + b_n \sin nx\right)$$

$$= \frac{\pi/2}{2} + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \pi/4 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\boxed{f(x) = \pi/4 - 3\sin x + \frac{3}{2}\sin 2x - \sin 3x + \frac{3}{4}\sin 4x + \dots}$$

Ans

Q#3:-

$$y'' - 4y' + 13y = 8\sin 3x \rightarrow (1)$$

$$y(0) = 1 \,,\; y'(0) = 2$$

Sol:- Associated Homogenous equ of (1) is

$$y'' - 4y' + 13y = 0 \rightarrow (2)$$

Change (2) into Auxiliary equation

Put $y = m$ in (2)

$$m^2 - 4m + 13 = 0$$

Use Quadratic formula

$$a = 1 \,,\; b = -4 \,,\; c = 13$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36}\, i}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= \boxed{2 \pm 3i}$$

$$\boxed{\begin{aligned} m_1 &= 2 + 3i \\ m_2 &= 2 - 3i \end{aligned}}$$

$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) \rightarrow$ (A)

Let $y_p = A\cos 3x + B\sin 3x \rightarrow$ (*)

Diff: with respect to "x"

$y'_p = -3A\sin 3x + 3B\cos 3x$

Again Diff: with respect to "x"

$y''_p = -9A\cos 3x - 9B\sin 3x$

Put in (1)

$\Rightarrow (-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x$

$+ 3B\cos 3x) + 13(A\cos 3x + B\sin 3x)$

$= 8\sin 3x$

$\Rightarrow -9A\cos 3x - 12B\cos 3x + 13A\cos 3x$

$-9B\sin 3x + 12A\sin 3x + 13B\sin 3x$

$= 8\sin 3x$

$$\Rightarrow (-9A - 12B + 13A)\cos 3x + (-9B + 12A + 13B)\sin 3x$$
$$= 8\sin 3x$$

$$\Rightarrow (4A - 12B)\cos 3x + (4B + 12A)\sin 3x = 8\sin 3x$$

Comparing Co-efficients

$\sin 3x \Rightarrow 4B + 12A = 8 \rightarrow$ ⓐ

$\cos 3x \Rightarrow 4A - 12B = 0 \Rightarrow 4A = 12B$

$\Rightarrow \boxed{A = 3B} \rightarrow$ ⓑ

Put ⓑ in ⓐ

$$4B + 12(3B) = 8$$
$$4B + 36B = 8$$
$$40B = 8$$
$$\boxed{B = 1/5} \rightarrow ①$$

Put © in (b)

$$\Rightarrow \boxed{A = 3/5} \rightarrow (d)$$

Put © and (d) in (*)

$$y_p = \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \rightarrow (B)$$

The General solution is

$$y = y_c + y_p$$

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{3}{5}\cos 3x$$
$$+ 1/5 \sin 3x \rightarrow ©$$

Now we need to find the values of c_1 and c_2 for this

Put $x = 0$ and $y = 1$ in ©

$$1 = e^{2(0)}\left(c_1 \cos 3(0) + c_2 \sin 3(0)\right) + \frac{3}{5}\cos 3(0)$$

$$+ \frac{1}{5}\sin 3(0)$$

$$1 = \left(c_1(1) + c_2(0)\right) + \frac{3}{5}(1) + \frac{1}{5}(0)$$

$$1 = c_1 + 3/5$$

$$c_1 = 1 - \frac{3}{5}$$

$$\boxed{c_1 = 2/5} \rightarrow (*\,*)$$

Diff: (C) with respect to "x"

$$y' = c_1\left(2e^{2x}\cos 3x - 3e^{2x}\sin 3x\right) + c_2\left(2e^{2x}\sin 3x +\right.$$

$$\left.3e^{2x}\cos 3x\right) - 6/5 \sin 3x + \frac{3}{5}\cos 3x \rightarrow (D)$$

Put $y' = 2$, $x = 0$ in (D)

$$y' = c_1\left(2e^{2x}\cos 3x - 3e^{2x}\sin 3x\right) + c_2\left(2e^{2x}\sin 3x\right.$$

$$\left.+ 3e^{2x}\cos 3x\right) - 6/5 \sin 3x + \frac{3}{5}\cos 3x$$

Put $y' = 2$, $x=0$

$$2 = C_1\left(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0)\right) + C_2\left(2e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0)\right) - 6/5\sin 3(0) + \frac{3}{5}\cos 3(0)$$

$$2 = C_1(2) + C_2(3) - 0 + 3/5$$

$$2 = 2C_1 + 3C_2 + 3/5$$

Put $C_1 = 2/5$

$$2 = 4/5 + 3C_2 + 3/5$$

$$2 = 7/5 + 3C_2$$

$$3C_2 = 2 - 7/5$$

$$3C_2 = 3/5$$

$$\boxed{C_2 = 3/15} \rightarrow (***)$$

Put $(**)$ and $(***)$ in ©

$$y = e^{2x}\left(\frac{2}{5}\cos 3x + \frac{3}{15}\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$$

$$y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3 + \frac{1}{5}\sin 3x$$

$$\boxed{y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x}$$

Required General Solution.

Q# 4:- Solve:-

$$(D^2 - DD')z = \cos x \cos 2y$$

Sol:- We have

$$(D^2 - DD')z = \cos x \cos 2y \rightarrow ①$$

The Associated Homogenous Equation

$$(D^2 - DD')z = 0$$

Put $D_z = m$

$$((m^2) - (m)(1)) = 0$$

$$m^2 - m = 0$$

$$m(m-1) = 0$$

$$m = 0 \quad , \quad m = 1$$

Thus $Z_c = \phi_1(y) + \phi_2(y+x) \rightarrow (2)$

For Particular Solution:-

Let

$$Z_p = \frac{1}{D^2 - DD'} \cdot \cos x \cos 2y$$

$$Z_p = \frac{1}{2} \cdot \frac{1}{D^2 - DD'} \left[\cos(x-2y) + \cos(x-2y)\right]$$

$$Z_p = \frac{1}{2}\left[\frac{1}{D^2-DD'}\left[\cos(x-2y) + \frac{1}{D^2-DD'}\left[\cos(x-2y)\right]\right]\right.$$

Using the Integeral

$$Z_p = \frac{1}{2}\left[\frac{1}{-1+2}\cos(x+2y) + \frac{1}{-1-2}\cos(x-2y)\right]$$

$$= 1/2\left[1\cos(x-2y) - 1/3\cos(x-2y)\right]$$

$$Z_p = \frac{1}{2}\cos(x+2y) - 1/6\cos(x-2y)$$

Now the General Solution is:-

$$Z = Z_c + Z_p$$

$$Z = f_1(y) + f_2(y+x) + \tfrac{1}{2}\cos(x+2y) - \tfrac{1}{6}\cos(x-2y)$$

$$\boxed{Z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \tfrac{1}{6}\cos(x-2y)}$$ Ans

The End