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Subject Differential Equation

Summer Examination

Final Term

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Q No 1.

Find the Fourier series representation of
 $f(t) = 1+t, -\pi \leq t \leq \pi$

Solution:

Using formula

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos t + \sum_{n=1}^{\infty} b_n \sin t \rightarrow \text{eq (1)}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+t) dt$$

$$a_0 = \frac{1}{2\pi} \left[t + \frac{t^2}{2} \right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi} \left(\pi - (-\pi) + \frac{\pi^2}{2} - \left(-\frac{\pi^2}{2} \right) \right)$$

$$a_0 = \frac{1}{2\pi} \left(2\pi + \frac{2\pi^2}{2} \right)$$

$$a_0 = \frac{1}{2\pi} (2\pi + \pi^2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) (\cos nt) dt$$

$$a_n = \frac{1}{\pi} \left((1+t) \frac{\sin nt}{n} \Big|_{-\pi}^{\pi} - \int \left(\frac{\sin nt}{n} \frac{d}{dt} (1+t) \right) \right)$$

$$a_n = \frac{1}{\pi} \left((1+t) \frac{\sin nt}{n} \Big|_{-\pi}^{\pi} - \frac{\cos nt}{n^2} \Big|_{-\pi}^{\pi} \right)$$

$$a_n = \frac{-1}{n^2 \pi} \left(\cos n \pi - \cos n(-\pi) \right)$$

$$a_n = \frac{-1}{n^2 \pi} (-1 - (-1))$$

$$\boxed{a_n = 0}$$

Now

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) \sin nt dt$$

$$b_n = \frac{1}{\pi} \left((1+t) \int_{-\pi}^{\pi} \sin nt - \int \left(\int \sin nt \frac{d}{dt} (1+t) \right) dt \right)$$

$$b_n = \frac{1}{\pi} \left((1+t) (-\cos nt) \Big|_{-\pi}^{\pi} - \int \left(-\frac{\cos nt}{n} (1) \right) \right)$$

$$b_n = \frac{1}{\pi} \left(\frac{-1(1+t)(\cos nt)}{n} \Big|_{-\pi}^{\pi} + \left(\frac{\sin nt}{n^2} \Big|_{-\pi}^{\pi} \right) \right)$$

$$b_n = \frac{-1}{n\pi} \left((1+\pi) (\cos n\pi) - (1+(-\pi)) (\cos n(-\pi)) \right)$$

$$b_n = \frac{-1}{n\pi} (\cos n\pi + \pi \cos n\pi - \cos \pi + \pi \cos \pi)$$

$$b_n = -\frac{1}{n\pi} (2\pi \cos n\pi)$$

$$\text{Here } \cos n\pi = \frac{(-1)^{n+1}}{n}$$

$$b_n = \frac{2}{\pi} (-1)^{n+1}$$

then eq become

$$f(t) = \frac{1}{2\pi} (2\pi + \pi^2) + 0 + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin t$$

Qno 2:-

Calculate the characteristic eq
the Eigen value of the system.
where A is given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix} = A$$

Eigen values = ?

Solution:-

We have

$A =$ Given Matrix

$$(A - \lambda I)x = 0$$

Now the characteristic equation is
given by

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 1-\lambda & 4 \\ 0 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - \left| \begin{array}{c} \text{Sum of dia-} \\ \text{gonal elements} \end{array} \right| \lambda^2 + \left| \begin{array}{c} \text{Sum of} \\ \text{diagonal} \\ \text{minor} \end{array} \right| \lambda - |A| = 0 \rightarrow \textcircled{B}$$

$$\text{Sum of Diagonal elements} = 1+1+2=4$$

$$\begin{aligned} \text{Sum of Diagonal minors} &= \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \\ &= (-6) + (2) + (1) \\ &= -6 + 2 + 1 \\ &= -3 \end{aligned}$$

By putting values in eq \textcircled{B}

$$\lambda^3 - 4\lambda^2 - 3\lambda - |A| = 0 \text{ --- } \textcircled{C}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \\ &= 1(2-8) - 0 + 1(6-0) \\ &= -6 + 6 \\ &= 0 \end{aligned}$$

Now putting values in \textcircled{C}

$$\lambda^3 - 4\lambda^2 - 3\lambda - 0 = 0$$

$$\lambda^3 - 4\lambda^2 - 3\lambda = 0$$

$$\lambda (\lambda^2 - 4\lambda - 3) = 0$$

$$\lambda = 0$$

$$\lambda^2 - 4\lambda - 3 = 0$$

Using Quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1$$

$$b = -4$$

$$c = -3$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$$

$$= 4 \pm \sqrt{\frac{16 + 12}{2}}$$

$$\lambda = 4 \pm \sqrt{\frac{28}{2}}$$

$$\lambda = 4 + \frac{\sqrt{28}}{2}, \quad \lambda = 4 - \frac{\sqrt{28}}{2}$$

we have eigen values

$$\lambda = \left(0, \frac{4 + \sqrt{28}}{2}, \frac{4 - \sqrt{28}}{2} \right)$$

Solution

Q.103:-

Solve the following system of linear equations.

$$5x + 0 + 4z + 2m = 3$$

$$x - y + 2z + m = 1$$

$$4x + y + 2z + 0 = 1$$

$$x + y + z + m = 0$$

Solution:

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right] \xrightarrow{R_4 R_2}$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 0 & 2 & -1 & 0 & -1 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 + \frac{4}{5} & 1 \\ 0 & -1 & +\frac{6}{5} & +\frac{4}{5} & \frac{3}{5} \\ 0 & 2 & -1 & 0 & -1 \end{array} \right] \xrightarrow{-\frac{1}{5} \times R_3}$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -1 & 6/5 & 4/5 & 3/5 \\ 0 & 0 & 7/5 & 8/5 & 1/5 \end{array} \right] \quad \begin{array}{l} \underbrace{5 \times R_3} \text{ and } \underbrace{5 \times R_4} \end{array}$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \begin{array}{l} \underbrace{5R_3} \text{ and } \underbrace{5R_4} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \frac{1}{5} \times R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -1 & 6/5 & 1/5 & 2/5 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \underbrace{R_2 \times 5}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \underbrace{R_3 - R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 8/7 & 1/7 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right] \quad \begin{array}{l} \underbrace{R_3 \leftrightarrow R_4} \\ \underbrace{1/7 \times R_3} \\ \underbrace{1/3 \times R_4} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right] \quad \begin{array}{l} \curvearrowright \\ C_2 \times -5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 6 & 1 & 2 \\ 0 & 0 & 1 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & -5 & 26/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1/2 & 3/4 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 6 & 1 & 1/3 \end{array} \right] \quad \begin{array}{l} \curvearrowright \\ 5/4 \times R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 126/84 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 3/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 3/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$(x, y, z, m) = (3/4, 3/21, -11/21, 1/3)$$

$$x = 3/4$$

$$y = 3/21$$

$$z = -11/21$$

$$m = 1/3$$

Ques 4:-

Verify that

$$u(x, t) = \sin(x + 2t)$$

is a solution of the one-dimensional wave equation.

Solution:-
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, t) = \sin(x + 2t)$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

it is satisfy the above eq

$$\frac{\partial u}{\partial t} = \cos(x + 2t) \cdot \frac{d}{dt}(x + 2t)$$

$$\frac{\partial^2 u}{\partial t^2} = -2 \sin(x + 2t) \cdot \frac{\partial}{\partial t}(x + 2t)$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = -4 \sin(x + 2t)} \rightarrow \textcircled{A}$$

Now

$$\frac{\partial u}{\partial x} = \cos(x+2t)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t)$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t)} \rightarrow \textcircled{B}$$

Comparing A and B

$$\boxed{c=2}$$

$$\Rightarrow -4\sin(x+2t) = -c^2\sin(x+2t)$$

$$\Rightarrow -4\sin(x+2t) + c^2\sin(x+2t) = 0$$

This is possible if $c = \pm 2$

$$-4\sin(x+2t) + (\pm 2)^2\sin(x+2t) = 0$$

$$\boxed{0=0}$$

This

$$y(x,t) = \sin(x+2t)$$