Differential Equations
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Final Term

Question # 01

a) Define 2nd order linear homogenous / non-homogenous differential equation along with examples.

Homogenous Differential Equation:-

That differential equation of any order is homogenous if once all the terms involving the unknown function are collected together on one side of the equation and the other side is identically zero.

Example :-

$$y'' - 2y' + y = 0$$

$$y'' + f(x)y' + g(x)y = r(x)$$

Non-Homogenous Differential Equations:

The non-homogenous differential equation has terms on both sides in this type of equation has the form of

$$y'' + Py' + qy = f(x)$$

where P, q are real no & can be real and complex.

Example:

$$y'' - 4y' + 3y = 2e^{-2x}$$

1 b) Solve the following 2^{nd} order linear homogenous / non-homogenous differential equation?

(i) $4y'' - 6y' + 7y = 0$

Solution:-

For an equation $ay' + by' + cy = 0$.

assume solution of form e^{xt} receiving equation with $y = e^{2}t$

$$4[(e^{xt})''] - 6[(e^{xt})] + 7(e^{xt}) = 0$$

$$e^{xt}[4x^2 - 6x + 7] = 0 \quad \text{———} \quad (1)$$

By solving (1)

$$\lambda = \frac{3}{4} + i\frac{\sqrt{19}}{4}, \quad \lambda = \frac{3}{4} - i\frac{\sqrt{19}}{4}$$

For 2 complex roots.

$$r_1 \neq r_2$$

where $r_i = a + i\beta$, $r_2 = a - i\beta$

The general solution is

$$y = e^{at}(C_1 \cos(\beta t) + C_2 \sin \beta t)$$

$$y'' - 4y' - 12y = 3e^{5x}$$

Solution:-

$$y = y_h + y_p$$

y_h is solution of homogenous ODE, by using

$$y'' - 4y' - 12y = 0$$

$$(D^2 - 4D - 12)\,y = 0$$

$$m^2 - 4m - 12 = 0$$

$$m = \frac{D^2 \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(D(-1))}}{2}$$

$$y_h = C_1 e^{6x} + C_2 e^{-2x}$$

y_p is that which satisfies

$$y'' - 4y' - 12y = 3e^{5x}$$

so,

$$y_p = \frac{-3}{7} e^{5x}$$

$$y = C_1 e^{6x} + C_2 e^{-2x} - \frac{3}{7} e^{5x}$$

Question # 02

Solve the following IVP for the 2^{nd} order linear equation.

(i) $16y'' - 40y' + 25y = 0 \qquad Y(0) = 3 \quad y'(0) = -9/4$

Solution:-

Let

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

Now

$$16r^2e^{rx} - 40re^{rx} + 25e^{rx}$$

$$e^{rx}(16r^2 - 40r + 25) = 0$$

$$16r^2 - 40r + 25 = 0$$

$$16r^2 - 20r - 20r + 25 = 0$$

$$4(4r - 5) - 5(4r - 5) = 0$$

So,

$$r_1 = 5/4 \quad , \quad r_2 = 5/4$$

$$y = e^{5/4x} \quad , \quad y = e^{5/4x}$$

General Solution

$$y = C_1 e^{5/4x} + C_2 e^{5/4x}$$

Satisfy ICs

$$y(0) = 3 : y(0) = C_1 e^{5/4(0)} + C_2 e(0)$$

$$= C_1 + C_2 = 3 \quad \text{—— (1)}$$

$$y'(0) = 5/4\, C_1 e^{5/4(0)} + C_2 e^0 = 5/4\, C_1 + C_2$$

Eq (1)

$$C_1 + C_2 = 3$$

Eq (2)

$$5/4 \, C_1 + C_2 = 9/4$$

$$\cancel{4} \times \frac{5}{\cancel{4}} C_1 + C_2 \times 4 = \frac{9}{\cancel{4}} \times \cancel{4}$$

$$5C_1 + 4C_2 = 9 \quad \text{——} (2)$$

Combining eq 1 & 2

$$C_1 = 3 - C_2$$

Put in Eq (2)

$$5(3 - C_2) + 4C_2 = 9$$

$$15 - 5C_2 + 4C_2 = 9$$

$$-C_2 = 9 - 15$$

$$\cancel{+} C_2 = + 6$$

$$\boxed{C_2 = 6}$$

$$C_1 + C_2 = 3$$

$$C_1 = 3 - C_2$$

$$C_1 = 3 - 6$$

$$\boxed{C_1 = -3}$$

So,

$$y = -3e^{5/4x} - 6e^{5/4x}$$

Q2 (ii)

$$y'' + 14y' + 49y = 0 \;,\; y(-4) = -1, y'(-4) = 5$$

Solution:-

Let

$$y' = re^{rx} \;,\; y'' = r^2 e^{rx}$$

$$r^2 e^{rx} + 14\, re^{rx} + 49e^{rx} = 0$$

$$e^{rx}(r^2 + 14r + 49) = 0$$

$$r^2 + 14r + 49 = 0$$

$$r^2 + 7r + 7r + 49 = 0$$

$$r(r+7) + 7(r+7) = 0$$

$$r_1 = 7 \;,\; r_2 = 7$$

$$y = e^{7x} \;,\; y = e^{7x}$$

General Solution

$$y = C_1 e^{7x} + C_2 e^{7x}$$

Satisfy ICs

$$y(-4) = 1,\; y(-4) = C_1 e^{7(0)} + C_2 e^{7(0)}$$

$$= e^{7}(C_1 + C_2) = -1$$

$$y'(-4) = 5 \;,\; y(-4) = 7C_1 e^{7(0)} + C_2 e^{7(0)}$$

$$= 7C_1 + C_2 = 5$$

eq 1

$$C_1 + C_2 = -1$$

eq 2

$$7C_1 + C_2 = 5$$

Now

$$C_2 = -1 - C_1$$

Put in (2)

$$7C_1 + (-1 - C_1) = 5$$

$$7C_1 - 1 - C_1 = 5$$

$$6C_1 - 1 = 5$$

$$\boxed{C_1 = 1}$$

and

$$\boxed{C_2 = -2}$$

So,

$$y = e^{7x} - 2e^{7x}$$

Q2 (iii)

$$y'' - 4y' + 9y = 0, \; y(0) = 0, y'(0) = -8$$

Solution:-

Let

$$y'' = r^2 e^{rx}$$

$$y' = re^{rx}$$

$$y = e^{rx}$$

$$r^2 e^{rx} - 4re^{rx} + 9e^{rx} = 0$$

$$e^{rx}(r^2 - 4r + 9) = 0$$

$$r^2 - 4r + 9 = 0$$

Apply quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(4)^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 36}}{2}$$

$$= \frac{4 \pm \sqrt{-20}}{2}$$

$$= 2 \pm \sqrt{-20}$$

$$= 2 \pm \sqrt{-20}$$

$$= 2 \pm \sqrt{-1 \cdot 20}$$

$$= 2 \pm \sqrt{-1 \cdot 45}$$

$$= 2 \pm i\sqrt{2^2 \cdot 5}$$

$$= 2 \pm i\, 4\sqrt{5}$$

$$y = e^{2+i4\sqrt{5}x} \quad , \quad y = e^{2-i4\sqrt{5}x}$$

General Solution.

$$y = C_1 e^{2+i4\sqrt{5}x} + C_2 e^{2-i4\sqrt{5}x}$$

Satisfy ICS

$$y(0) = 0 , \; y(0) = C_1 e^{2+i4\sqrt{5}x} (0) + C_2 e^{\circ}$$

$$= C_1 + C_2 = 0$$

$$y'(0) = -8 , \; y'(0) = C_1 e^{2+i4\sqrt{5}x} (0) + C_2 e^{0}$$

$$= 2 + i\, 4\sqrt{5}x \; C_1 + C_2$$

$$= -8$$

$$C_1 + C_2 = 0$$

$$C_1 = -C_2$$

$$2i\, 4\sqrt{5}x \; C_1 + C_2 = 8$$

$$2i\, 4\sqrt{5}x \; C_1 + C_1 = 8$$

$$2i\, 4\sqrt{5}x \; 2C_1 = 8$$

$$2C_1 = \frac{8}{2i4\sqrt{5}x}$$

$$C_1 = \frac{16}{2i\sqrt{5}x}$$

$$C_2 = \frac{-16}{2i\sqrt{5}x}$$

$$y = \frac{16}{2i\sqrt{5}x} e^{2+i4\sqrt{5}x} - \frac{16}{2i\sqrt{5}x} e^{2-i4\sqrt{5}x}$$

Q2 (iv)

$$y'' - 8y' + 17y = 0, \quad y(0) = -4, \; y'(0) = -1$$

Solution:-

$$r^2 e^{rx} - 8re^{rx} + 17e^{rx} = 0$$

$$e^{rx}(r^2 - 8r + 17) = 0$$

$$r^2 - 8r + 17 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-8) \pm \sqrt{(8)^2 - 4(1)(7)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{16 - 68}}{2}$$

$$= 4 \pm \sqrt{-52}$$

$$= 4 \pm \sqrt{-1 \cdot \times 52}$$

$$= 4 \pm i\sqrt{04 \cdot 13}$$

$$= 4 \pm 2i\sqrt{13}$$

$$y = 4 + 2i\sqrt{13}, \quad y = 4 - 2i\sqrt{13}$$

$$y = C_1 e^{4+2i\sqrt{13}}, \quad y = c_1 e^{4+2i\sqrt{13}}$$

$$y(0) = -4, \quad y(0) = C_1 e^{4+2i\sqrt{3}}(0) + C_2 e^0$$

$$= C_1 + C_2 = -4$$

$y'(0) = -1, \; y'(0) = 4 + 2i\sqrt{3}\, c_1 e^{4+2i\sqrt{3}(0)} + C_2 e^0$

$= 4 + 2i\sqrt{3} C_1 + C_2$

$C_1 + C_2 = -4$

$C_2 = -4 - C_1$

Now

$4 + 2i\sqrt{3}\, C_1 + C_2$

$= 4 + 2i\sqrt{3} C_1 + (-4 - c_1)$

$= 4 + 2i\sqrt{3}\, c_1 - 4 + C_1$

$0 = 4 + 2i\sqrt{3}\; 2C_1 - 4$

$4 = 2C_1$

$\boxed{C_1 = 2}$

$C_2 = -4 - 2$

$C_2 = -6$

$y = 2e^{4+2i\sqrt{13}} - 6e^{4-2i\sqrt{13}}$

Question # 03

Define Laplace transform along with examples?

Laplace Transform :-

The Laplace Transform of a a signal (function) f is the function $F = \mathcal{L}(f)$ defined by

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

for those $s \in \mathbb{C}$ for which the integral makes sense.

- F is a complex-valued function of complex number.
- s is called the (complex) frequency variable, with units sec^{-1}; t is called the time variable (in sec); st is unitless.
- for now, we assume f contains no impulses at $t=0$.

Examples :-

$$F(s) = \int_0^{\infty} e^{-st} dt = \frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

$$f(t) = 1, \forall t \geq 0, \quad L[1] = \int_0^{\infty} e^{-st} dt = \frac{e^{-st}}{-s} \Big|_{t=0}^{t=\infty} = \frac{1}{s} = f(s)$$

for $s > 0$

$$F(s) = \int_0^{\infty} e^{t} e^{-st} dt = \int_0^{\infty} e^{(1-s)t} dt = \frac{1}{1-s} e^{(1-s)t} \Big|_0^{\infty} = \frac{1}{s-1}$$

Q3 (A)

Find the Laplace transform of the given functions.

1. $f(t) = 6\left[e^{-5t}\right] + e^{3t} + 5\left[t^{3}\right] - 9$

Solution:-

$$F(s) = \int_0^\infty 6e^{-5t}\, e^{st}\, dt + \int_0^\infty e^{-st}\, e^{at}\, dt + 5\int_0^\infty e^{-st} + dt$$

$$- \int_0^\infty e^{-st}\, 9dt$$

$$= 6\int_0^\infty e^{-(s+5)t}\, dt + \int_0^\infty e^{-(s-s)t}\, dt + 5\mathcal{L}\, t^{3} - \mathcal{L}9$$

$$= 6\left(0 - \frac{1}{-e^{3}(s+5)}\right) + 0 - \frac{1}{5(s-5)}\Big|_0^\infty + \frac{5\cdot 3!}{s^{3+1}} - \int_0^\infty e^{-st}\, 9t$$

$$= \frac{6}{s+5} + \frac{1}{5\cdot 3} + \frac{30}{s^{4}} - 9\left(\frac{e^{-st}}{-5}\Big|_0^\infty\right.$$

$$= \frac{6}{s+5} + \frac{1}{5\cdot 3} + \frac{30}{s^{4}} + 9\left(\frac{-1}{e^{0}s}\right)$$

$$= \frac{6}{s+5} + \frac{1}{5\cdot 3} + \frac{30}{s^{4}} - \frac{9}{5}$$

Answer.

Q3(A)

2) $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

Solution:-

$$= 4\,\mathcal{L}\{\cos 4t\} - 9\,\mathcal{L}\{\sin 4t\} + 2\,\mathcal{L}\{\cos 10t$$

$$= 4\left(\frac{s}{s^2+16}\right) - 9\left(\frac{4}{s^2+16}\right) + 2\left(\frac{s}{s^2+100}\right)$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

$$= \frac{4s-36}{s^2+16} + \frac{2s}{s^2+100}$$

3) $h(t) = e\hat{\ }3t + \cos(6t) - e\hat{\ }(3t)\cos(6t)$

Solution:-

$$h(t) = \mathcal{L}\{e^{3t}\} + \mathcal{L}\{\cos 6t\} - \mathcal{L}\{e^{3t}\cos 6t\}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

$$= \frac{1}{(s-3)} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

Question #04

Solve the following IVP using Laplace Transform.

(i) $y'' - 10y' + 9y = 5t, \quad y(0) = -1, \quad y'(0) = 2$

Solution:-

Applying the Laplace transform to both sides, we find

$$(s^2 - 10s + 9)Y + s - 2 - 10 = \frac{5}{s^2} \Rightarrow Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

To find inverse Laplace transform we will need simplify the expression for $Y(s)$ using the partial fraction decomposition.

$$\frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

We find

$$B = \frac{5}{9}, \quad D = -2, \quad C = \frac{31}{81}, \quad A = \frac{50}{81}$$

Therefore, using linearity of the inverse Laplace transform

$$y(t) = \frac{50}{81} + \frac{5t}{9} + \frac{31}{81}e^{9t} - 2e^t$$

Q4 (ii)

$$y'' - 6y' + 15y = 2\sin(3t), \quad y(0) = -1 \quad y'(0) = -4$$

Solution:-

We have

$$(s^2 - 6s + 15)Y + s - 2 = \frac{6}{s^2+9} \Rightarrow Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}$$

To find the Constants, we need to simplify the expression

$s^3 : A + C = -1$

$s^2 : -6A + B + D = 2$

$s^1 : 15A - 6B + 9C = -9$

$s^0 : 15B + 9D = 24$

The solution is

$$A = \frac{1}{10}, B = \frac{1}{10}, C = \frac{11}{10}, D = \frac{5}{2}$$

Hence we got

$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15}\right)$$

Now we need to find inverse laplace transform.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{3}\mathcal{L}^{-1}$$

$$\left\{\frac{3}{s^2+9}\right\} = \cos 3t + \frac{1}{3}\sin 3t$$

The second term is slightly more involved.

$$\frac{-11s+25}{s^2-6s+15} = \frac{-11s+25}{(s-3)^2+6}$$

$$= \frac{-11(s-3)-8}{(s-3)^2+6}$$

$$= -11 \frac{(s-3)}{(s-3)^2+6} - \frac{8}{\sqrt{6}} \frac{\sqrt{6}}{(s-3)^2+6}$$

Now

$$\mathcal{L}^{-1}\left\{\frac{-11s+25}{s^2-6s+15}\right\} = -11\, e^{3t} \cos\sqrt{6}t - \frac{8}{\sqrt{6}} e^{3t} \sin\sqrt{6}t$$

The final answer hence is

$$y(t) = \mathcal{L}^{-1}\{Y\} = \frac{1}{10}\left(\cos 3t + \frac{1}{3}\sin 3t - 11\, e^{3t} \cos\sqrt{6}t - \frac{8}{\sqrt{6}} e^{3t} \sin\sqrt{6}t\right)$$