Department of Computer Science

Date: 13th April 2020

Midterm Assignment –Spring 2020

Course Title: Differential Equations

Instructor: Engr. Latif Jan

Program: BS (CS-SE-EE) **Total Marks:** 30 **Time Allowed:** 6 days

Note: Attempt all Questions:

Q 1: a) Define differential equation along with 2 examples? **(1+1 Marks)**

b) Define a Separable Differential Equation (DE)? **(1+4+3 Marks)**

i. Solve the following **Initial Value Problem (IVP) using separable DE** and find the interval of validity of the solution.

$$\text{(a) } y' = \frac{xy^3}{\sqrt{1+x^2}} \quad y(0) = -1$$

$$\text{(b) } y' = e^{-y}(2x-4) \quad y(5) = 0$$

Q 2: a) Solve the following IVP using Linear Differential method **(2+5+3 Marks)**

(i) **Explain the steps for solving Linear Differential Equation.**

(ii) $cos(x)y' + sin(x)y = 2cos^3(x)sin(x) - 1 \quad y\left[\frac{\pi}{4}\right] = 3\sqrt{2}, \quad 0 \le x \le \frac{\pi}{2}$

(iii) $x' + 2x = sint$

Q 3: Solve the following IVP for the exact equation and find the interval of validity for the solution. **(5+5 Marks)**

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = -3$

(ii) $\frac{2ty}{t^2+1} - 2t - (2 - ln(t^2+1))y' = 0 \quad y(5) = 0$

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Program - B(E)E

Course title : Differential Equation.

Task :- Mid Term Assignment Spring 2020

Submitted To : Engr Latif Jan

Q1.(a) Define Differential Equation along with 2 Examples ?

Ans:- **Differential Equation:-**

A differential Equation is an Equation that relates one or more Functions and Their derivatives.
The derivatives represent Their rates of change and The differential Equation defines a relationship between The two.

For Example:-

(i) $\frac{dy}{dx} = f(x)$

Here "x" is a dependent variable and "y" is independent.

(ii) $\frac{dy}{dx} = 5x$

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(b) Define a Separable Differential Equation (DE)?

Ans:- A differential equation is said to be separable if the variable can be Seperated.

$$F(Y)\,dY = G(x)\,dx.$$

Q1(i):- Solve the following initial value problem using Separable DE and find the interval of valadity of the solution.

(a) $Y' = \dfrac{xy^3}{\sqrt{1+x^2}}$ $\quad Y(0) = -1$

③

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

Sol:-

Seperate :

$$y^{-3} dy = x(1+x^2)^{-\frac{1}{2}} dx.$$

integrate

$$\int y^{-3} dy = \int x(1+x^2)^{-\frac{1}{2}} dx.$$

$$\frac{y^{-3+1}}{-3+1} = \sqrt{1+x^2} + c$$

$$\frac{y^{-2}}{-2} = \sqrt{1+x^2} + c$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

Now Apply The initial condition to get The Value of c

$$-\frac{1}{2} = \sqrt{1} + c \qquad c = -\frac{3}{2}$$

The implicit Solution is Then

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now Solve for $y(x)$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

$$y(x) = \pm \frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

So the initial condition shows '–'

Then

$$\boxed{y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}}$$

To get the interval of validity

Since $\because 1+x^2 \geq 0$. ∵ (division by zero) '–ve' under the outer root.

$$3-2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4(1+x^2)$$

$$\frac{9}{4} > 1+x^2$$

$$\frac{5}{4} > x^2$$

Finally solving for 'x' (square root of negative number will be)

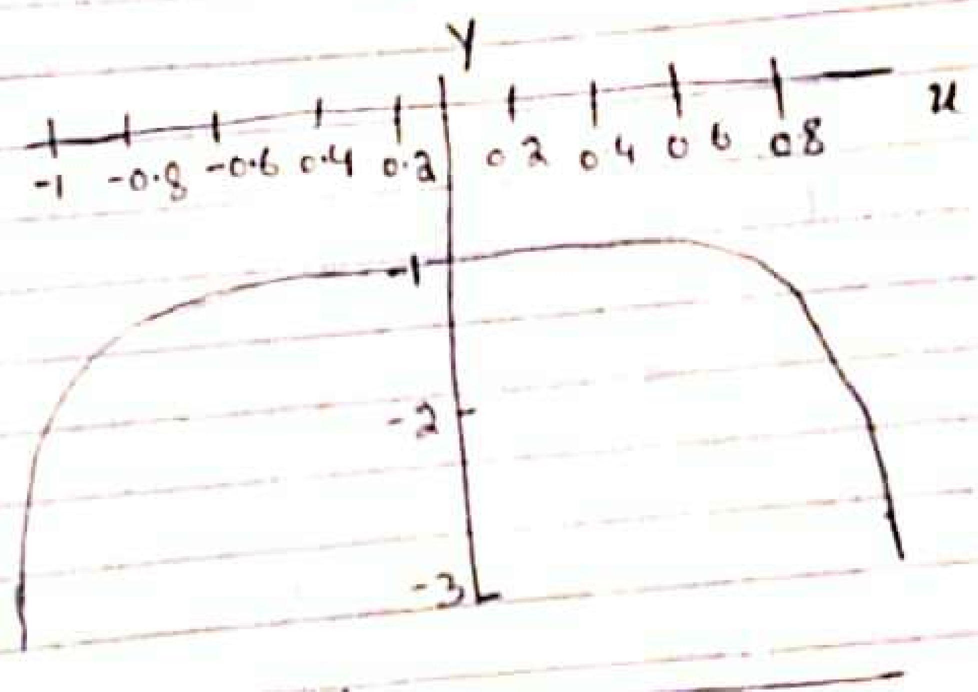
$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

The initial condition

$x = 0$

This interval is Therefore our interval of validity.

So graph of solution.



X

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(b) $y' = e^{-y}(2x-4)$

$y(5)=0$.

Solution:-

Seperate and integrate.

$e^{y}dy = (2x-4)dx$

Now integrate

$\int e^{y}dy = \int(2x-4)\,dx$.

$= \cancel{2}\frac{x^2}{\cancel{2}} - 4x + c$

$e^{y} = x^2 - 4x + c$

Apply The initial condition

$1 = 25 - 20 + c \qquad c = -4$

gives implicit solution of

$e^{y} = x^2 - 4x - 4$

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By different equation
Taking log on b.s.

$$y(x) = \ln(x^2 - 4x - 4)$$

interval of valadity

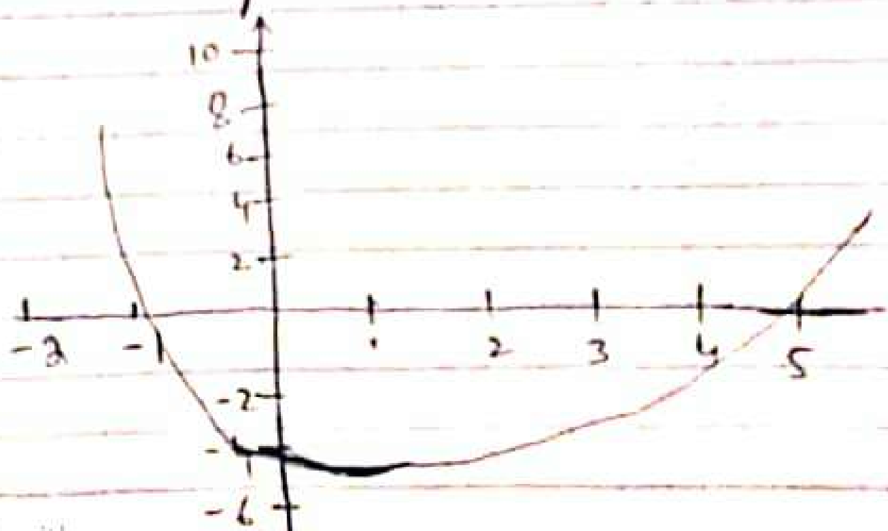
$$\boxed{x^2 - 4x - 4 > 0}$$

So

Qudratic will be zero at The two Points

$$x = 2 \pm 2\sqrt{2}$$

So graph of quadratic shows



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So possible interval

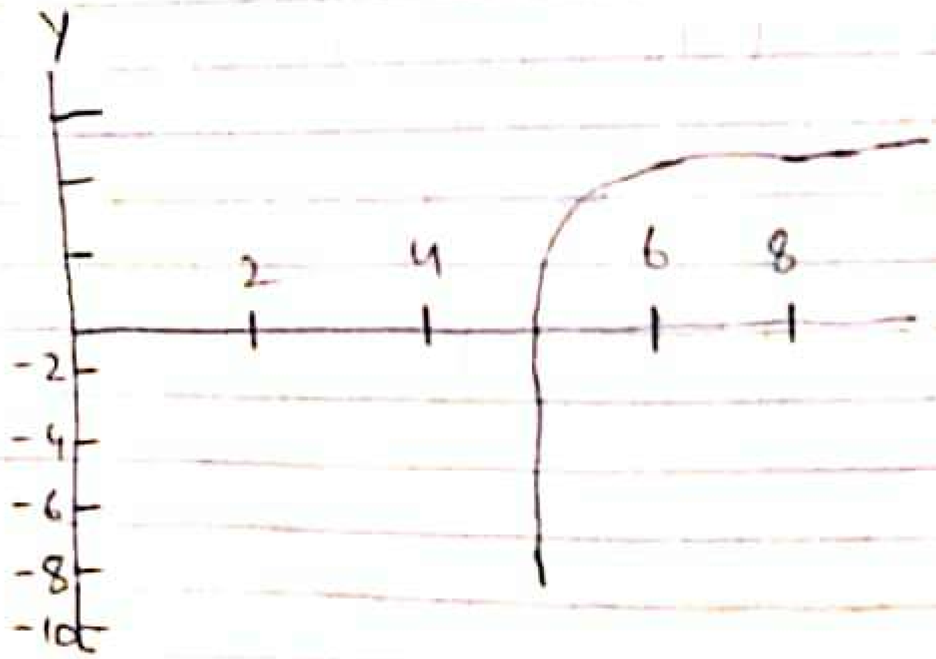
$$\infty < x < 2 - 2\sqrt{2}$$

$$2 + 2\sqrt{2} < x < \infty$$

$x = 5$

$$2 + 2\sqrt{2} < x < \infty$$

Graph of Solution



Q:- Solve The following IVP for The exact equation and find The interval of validity for The solution.

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$

$y(0) = -3$

Solution :-

First identify M and N.

$M = 2xy - 9x^2 \qquad M_y = 2x$

$N = 2y + x^2 + 1 \qquad N_x = 2x$

Now we have to find

$\psi(x, y) = ?$

$\psi_x = M$

$\psi_y = N$

By integrating

$$\psi = \int M\,dx \quad \text{OR} \quad \psi = \int N\,dy$$

for First one.

$$\psi(x,y) = \int 2xy - 9x^2\,dx =$$
$$x^2y - 3x^3 + h(y)$$

$$\psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$$

$$h'(y) = 2y + 1$$

Find $h(y)$ by integrating

$$h(y) = \int 2y + 1\,dy = y^2 + y + K$$

$$\psi(x,y) = x^2y - 3x^3 + y^2 + y + K =$$
$$y^2 + (x^2+1)y - 3x^3 + K$$

(12)

$$y^2 + (x^2+1)\,y - 3x^3 + K = C$$

$$y^2 + (x^2+1)\,y - 3x^3 = C - K$$

$$y^2 + (x^2+1)\,y - 3x^3 = c$$

Apply The initial condition

put The value of c.

$$(-3)^2 + (0+1)(-3) - 3(0)^3 = c$$

$$\Rightarrow \quad c = 6$$

$$y^2 + (x^2+1)\,y - 3x^3 - 6 = 0$$

By Quadratic formula

$y(x) = ?$

$$y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^3-6)}}{2(1)}$$

$$= \frac{-(x^2+1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}$$

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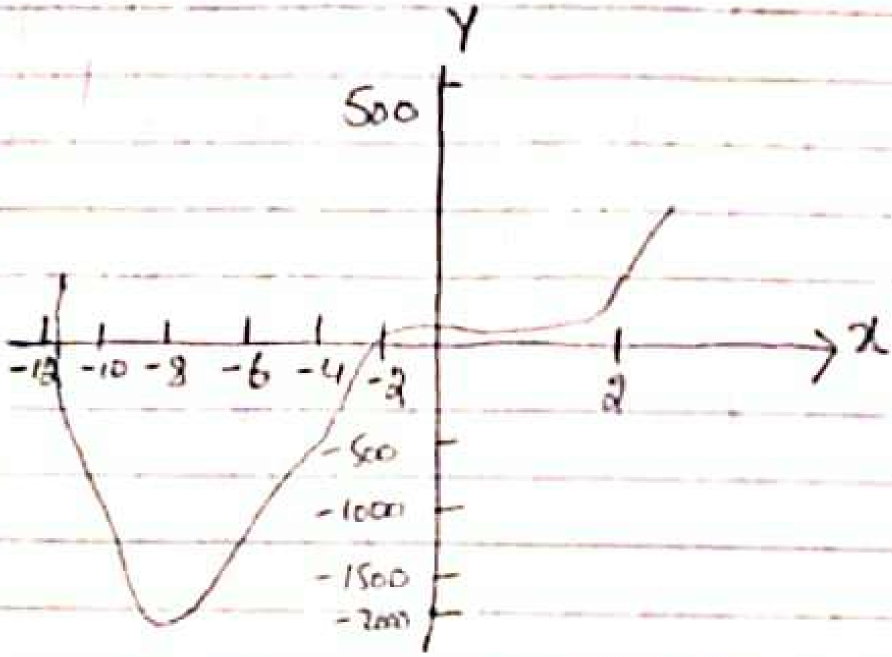
$$x^4+12x^3+2x^2+25=0$$

So The graph Of The Polynomial Under The radical



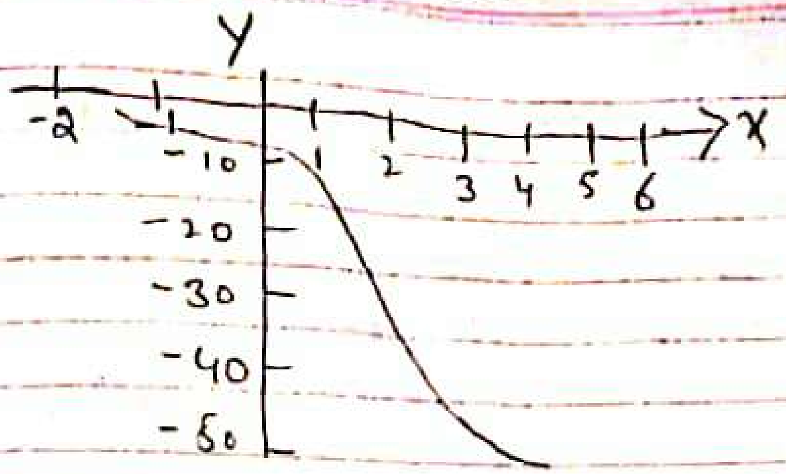
There are Two intervals where The Polynomial will be positive.

$$-\infty < x \leq -11.81557624$$

$$-1.39691133 \leq x < \infty$$

Validity must be

$$-1.3969111 33 \leq x < \infty$$



Q (ii) $\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1)) y' = 0$

$y(5) = 0$

Solution

Separating Minus Sign of two terms:

$$\frac{2ty}{t^2+1} - 2t + \left(\ln(t^2+1) - 2\right) y' = 0$$

Now Find M and N

$M = \frac{2ty}{t^2+1} - 2t$ $\qquad M_y = \frac{2t}{t^2+1}$

$N = \ln(t^2+1) - 2$ $\qquad N_t = \frac{2t}{t^2+1}$

Integrate the first one.

$$\psi(t,y) = \int \frac{2ty}{t^2+1} - 2t\,dt = y\ln(t^2+1) - t^2 + h(y)$$

Differentiate with respect to y and compare to N

$$\psi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2 = N$$

so $h'(y) = -2 \Rightarrow h(y) = -2y$

$$\psi(t,y) = y\ln(t^2+1) - t^2 - 2y$$

Implicit solution is then

$$y\ln(t^2+1) - t^2 - 2y = c$$

Applying initial condition

$$-25 = c$$

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So

$$y\left(\ln(t^2+1)-2\right)-t^2=-25$$

Solve For y

$$y(t)=\frac{t^2-25}{\ln(t^2+1)-2}$$

$\because$

$\ln(t^2+1)-2=0$

$\ln(t^2+1)=2$

$t^2+1=e^2$

$t=\pm\sqrt{e^2-1}$

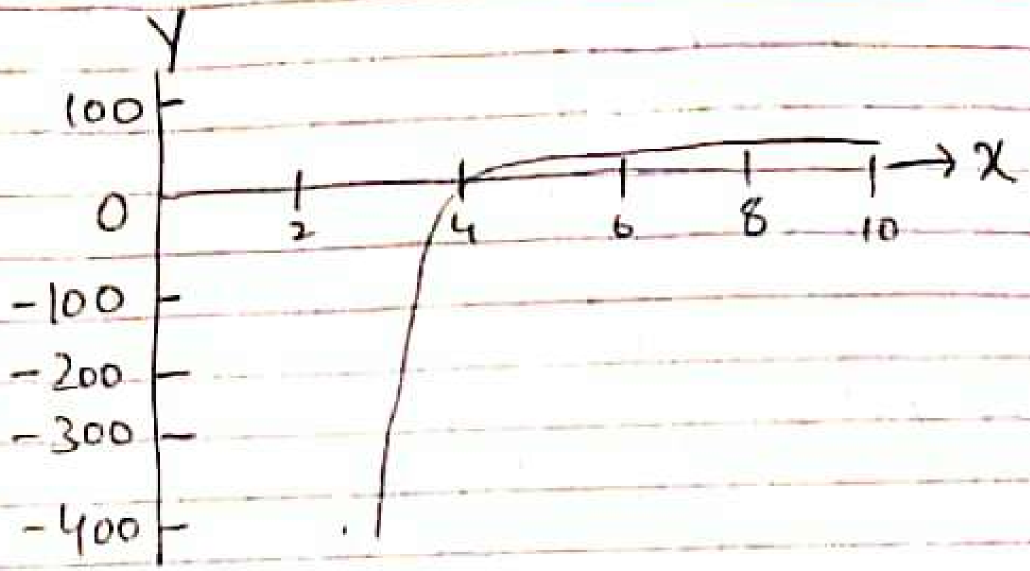
Now we have three Possible intervals

$$-\infty<t<-\sqrt{e^2-1}$$

$$-\sqrt{e^2-1}<t<\sqrt{e^2-1}$$

$$\sqrt{e^2-1}<t<\infty$$

So graph of Solution.



Q8- Solve The Following IVP Using Linear Differential Method.

(i) Explain The Steps for Solving linear Differential Equation.

Ans:- First order.

They are "First order"

When There is only

$$\frac{dy}{dx} \text{ not } \frac{d^2y}{dx^2} \text{ or } \frac{d^3y}{dx^3}$$

Linear:- A first order differential equation is linear when it can be made to look like this

$$\frac{dy}{dx} + P(x)y = Qx$$

Where $P(x)$ and $Q(x)$ are Function of x.

Steps:-

Substitute $y = uv$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

into

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- Factor the parts involving v.
- Solve using separation of variables to find u.
- Solve that to find v.
- Finally substitute u and v into $y = uv$ to get our solution.

(ii) $\cos x (y)' + \sin(x) y = 2\cos^3(x)\sin(x) - 1$

$$y\left[\frac{\pi}{4}\right] = 3\sqrt{2}$$

$$0 \le x \le \frac{\pi}{2}$$

Solution:- GET the coefficient

$$y' + \frac{\sin x}{\cos x} y = 2\cos^2 x \sin x - \frac{1}{\cos x}$$

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$y' + \tan(x)\,y = 2\cos^2(x)\sin(x)$
$- \sec(x)$

Now Find integrating factor:

$\mu(t) = e^{\int \tan(x)\,dx} = e^{\ln|\sec(x)|}$
$= e^{\ln \sec(x)} = \sec(x)$

$\int \tan(x)\,dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1}$
$= \ln|\sec(x)|$

$\because \boxed{e^{\ln f(x)} = f(x)}$

So
$\sec(x)\,y' + \sec(x)\tan(x)\,y) =$
$2\sec(x)\cos^2(x)\sin(x)$
$-\sec^2(x)$

$(\sec(x)\,y)' = 2\cos(x)\sin(x)$
$-\sec^2(x)$

integrate b.s.

$\int (\sec(x)\,y(x))'\,dx = \int 2\cos(x)$
$\sin(x) - \sec^2(x)\,dx$

$$\sec(x)\,y(x) = \int \sin(2x) - \sec^2(x)\,dx$$

$$\sec(x)\,y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + C$$

By using the trig formula

$$\sin(2\theta) = 2\sin\theta\cos\theta.$$

$$y(x) = -\frac{1}{2}\cos x\cos 2(x) - \cos(x)\tan(x) + C\cos(x)$$

$$= -\frac{1}{2}\cos(x)\cos 2(x) - \sin(x) + C\cos(x)$$

Find value of C

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\pi/2 - \sin(\pi/4) + C\cos(x)$$

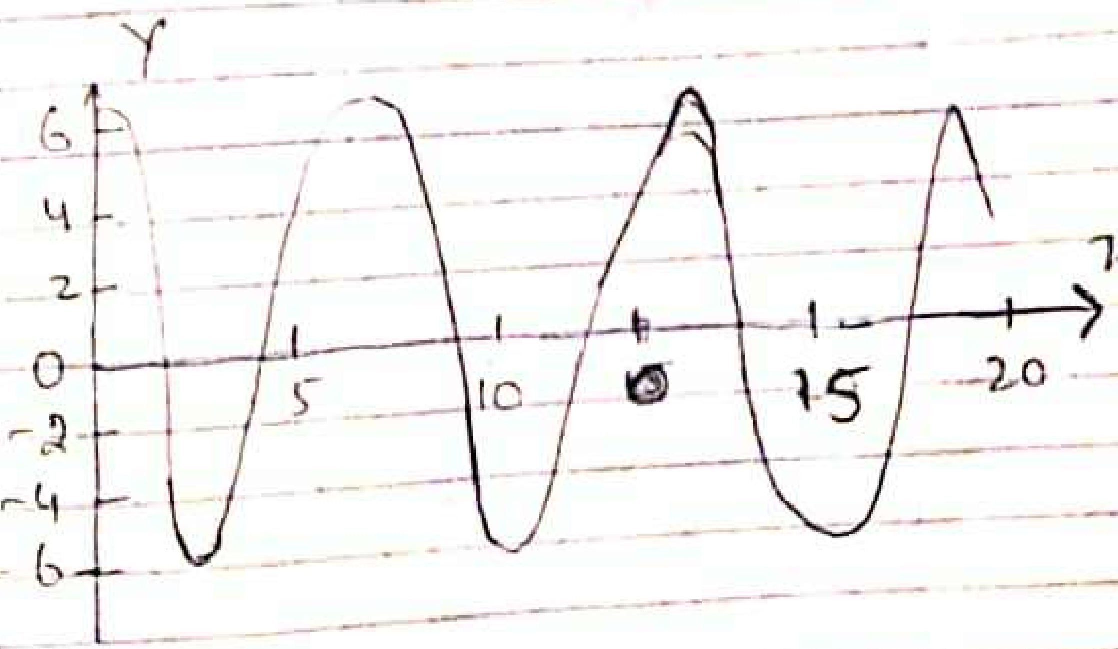
$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\,\frac{\sqrt{2}}{2}$$

$$C = 7$$

So

$$y(x) = -\frac{1}{2}\cos(x)\cos 2(x) - \sin(x) + 7\cos x.$$

Plot of The Solution.



(iii) $x' + 2x = \sin t$

Solution: $x' + 2x = \sin t$

$x' + \frac{1}{2}x = \sin t$

Find $y(t)$

$$\left(e^{+\frac{1}{2}x}\right)' = e^{\sin t}$$

Now integrate:

$$e^{+\frac{1}{2}x} = \int e^{\sin t}\,dt + c$$

$$e^{\frac{1}{2}x} = e^{\cos t} + c$$

$$y(t) = e^{\cos t} + c$$

Range of c

$c < 0$	$y(t) \to -\infty$
$c = 0$	$y(t)$ remain Finite
c	$t \to \infty$