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Linear Algebra

Semester # II

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BS - SE

Date: 10 June, 2020

Q1:-

Adjoint :

i)
$$A = \begin{bmatrix} 1 & 2 & 2nd\text{-}ID \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Sol:-

As:

$$A = \begin{bmatrix} 1 & 2 & 2nd\text{-}ID \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

put : ID-2nd = 5

∴ we get;
$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Now; Matrix A adjoint :

Adjoint Square matrix A is $(\text{cofactors}(A))^T$.

∴ Minors and cofactors for; $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$:

* Minors :- A Minor $M_{i,j}$ is the determinent of the square matrix formed by deleting Row i and column j from larger square matrix A.

* Co-factors :- A co-factor $C_{i,j}$ is Minor $+M_{i,j}$ when $i+j$ is even, and Minor $-M_{i,j}$ when $i+j$ is odd, thus ; $C_{i,j} = (-1)^{i+j} M_{i,j}$

Now:

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i.e.:

$M_{1,1}$: $\det\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = 6 - 1 = 5$

$M_{1,2}$: $\det\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = 4 - 3 = 1$

$M_{1,3}$: $\det\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} = 2 - 9 = -7$

$M_{2,1}$: $\det\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix} = 4 - 5 = -1$

$M_{2,2}$: $\det\begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix} = 2 - 15 = -13$

$M_{2,3}$: $\det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = 1 - 6 = -5$

$M_{3,1}$: $\det\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} = 2 - 15 = -13$

$M_{3,2}$: $\det\begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix} = 1 - 10 = -9$

$M_{3,3}$: $\det\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = 3 - 4 = -1$

$\therefore$ Matrix of Minors $M_{i,j}$

is $\begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{pmatrix} = \begin{bmatrix} 5 & 1 & -7 \\ -1 & -13 & -5 \\ -13 & -9 & -1 \end{bmatrix}$

Also; Matrix of Co-factors;

$$C_{i,j} = (-1)^{i+j} \cdot M_{i,j}$$

$$\therefore \begin{pmatrix} (-1)^{1+1} \cdot 5 & (-1)^{1+2} \cdot 1 & (-1)^{1+3} \cdot (-7) \\ (-1)^{2+1} \cdot (-1) & (-1)^{2+2} \cdot (-13) & (-1)^{2+3} \cdot (-5) \\ (-1)^{3+1} \cdot (-13) & (-1)^{3+2} \cdot (-9) & (-1)^{2+2} \cdot (-1) \end{pmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -7 \\ 1 & -13 & 5 \\ -13 & 9 & -1 \end{bmatrix}$$

Now;

Take Transpose of <u>Matrix of Co-factors</u>

i.e:

$$\begin{bmatrix} 5 & -1 & -7 \\ 1 & -13 & 5 \\ -13 & 9 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & 1 & -13 \\ -1 & -13 & 9 \\ -7 & 5 & -1 \end{bmatrix}$$

So;

adjoint of matrix A is;

$$\text{adj } A = \begin{bmatrix} 5 & 1 & -13 \\ -1 & -13 & 9 \\ -7 & 5 & -1 \end{bmatrix}$$

<u>Result</u>

Σ

(ii) $B = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 8 \end{bmatrix}$

Sol:- As; $B = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 8 \end{bmatrix}$; Adj $B \Rightarrow (\text{co-f}(B))^T$

∴ Finding Minors and cofactors;

i-e: $B = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 8 \end{bmatrix}$

$M_{i,j}$ of $B = \begin{bmatrix} 8 & -24 & 1 \\ 42 & -1 & -26 \\ 37 & 14 & -11 \end{bmatrix}$

Also;

$C_{i,j}$ of $B = \begin{bmatrix} (+1)(8) & (-1)(-24) & (+1)(1) \\ (-1)(42) & (+1)(-1) & (-1)(-26) \\ (+1)(37) & (-1)(14) & (+1)(-11) \end{bmatrix}$

$\Rightarrow C_{i,j}\ B = \begin{bmatrix} 8 & 24 & 1 \\ -42 & -1 & 26 \\ 37 & -14 & -11 \end{bmatrix}$

Take Transpose of $C_{i,j}$ B.

$\Rightarrow (C_{i,j}\ B)^T = \begin{bmatrix} 8 & 24 & 1 \\ -42 & -1 & 26 \\ 37 & -14 & -11 \end{bmatrix}^T$

$$\Rightarrow \text{Adj } B = \begin{bmatrix} 8 & -42 & 37 \\ 24 & -1 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

Result

Q2:- Co-factors of A_{21}, A_{31}, A_{33}.

i.e: $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

Sol:

As: $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

Finding Minors and Then Co-factors;

$$\therefore M_{i,j}(A) = \begin{bmatrix} 9 & -8 & -6 \\ 5 & -10 & 5 \\ -11 & 7 & -1 \end{bmatrix}$$

A_{21} A_{31} A_{33}

Now;

co-factors of; • $A_{21} = \left((-1)^{2+1} \cdot (5)\right) = (-1)^3 \cdot (5)$

$\Rightarrow \boxed{A_{21} = -5}$

• $A_{31} = \left((-1)^{3+1} \cdot (-11)\right) = (-1)^4 \cdot (-11)$

$\Rightarrow \boxed{A_{31} = -11}$, • $A_{33} = \left((-1)^{3+3} \cdot (-1)\right) = (-1)^6 \cdot (-1)$

$\Rightarrow \boxed{A_{33} = -1}$

Q3:- Eigen values and Eigen Vectors;

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} \quad \& \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol:

As;

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix},$$

* Matrix Eigenvectors;

Let A be a square matrix, η a vector and λ a scalar that satisfy: $A\eta = \lambda\eta$, then η is an eigen vector of A and λ is the eigenvalue associated with it.

Now,

to find η eigen vectors, we need to solve $\boxed{(A - \lambda I)\eta = 0}$ for each eigen-value;

Eigen values for $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$

Matrix Eigen values;

Let A is a square matrix, η a vector and λ a scalar that satisfy $A\eta = \lambda\eta$, then λ is called the eigen values associated with the eigen vector η of A.

- The Eigen values of A are the roots of the characteristic equation $\det(A - \lambda I) = 0$.

Now;

$$\det\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0 \ ?$$

so;

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1-0 & 1-0 \\ 1-0 & 3-\lambda & 2-0 \\ -1-0 & 1-0 & 2-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 2 \\ -1 & 1 & 2-\lambda \end{bmatrix}$$

$$\therefore \det\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 2 \\ -1 & 1 & 2-\lambda \end{bmatrix}$$

, Expanding by Row - 1.

$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} - (1)\begin{vmatrix} 1 & 2 \\ -1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ -1 & 1 \end{vmatrix}$$

$$= (2-\lambda)\Big((3-\lambda)(2-\lambda) - 2\Big) - (1)\Big((2-\lambda) - (-2)\Big) + (1)\Big(1 - (-1)(3-\lambda)\Big)$$

$$= (2-\lambda)\left(6 - 3\lambda - 2\lambda + \lambda^2 - 2\right) - (1)\left(2 - \lambda + 2\right) + (1)\left(1 + 3 - \lambda\right)$$

$$= (2-\lambda)(\lambda^2 - 5\lambda + 4) - (1)(-\lambda + 4) + 1(-\lambda + 4)$$

$$= 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda + \cancel{\lambda} - \cancel{4} - \cancel{\lambda} + \cancel{4}$$

$$= -\lambda^3 + 7\lambda^2 - 14\lambda + 8$$

Now;

$$-\lambda^3 + 7\lambda^2 - 14\lambda + 8 = 0$$

Solve, by factoring;

So;

$$-\lambda^3 + 7\lambda^2 - 14\lambda + 8 = 0$$

$$\Rightarrow \quad -1\left(\lambda^3 - 7\lambda^2 + 14\lambda - 8\right) = 0 \longrightarrow \text{(\#)}$$

we will use the "Rational Root Theorem"

to figure out factors;

i-e:

$$a_0 = 8 \quad , \quad a_n = 1$$

The dividers of a_0 : 1, 2, 4, 8

Also; divider of a_n : 1.

There, check following Rational numbers;

$\pm \frac{1, 2, 4, 8}{1}$, $\frac{1}{1}$, is a root of the expression, so factor out $(\lambda - 1)$

$$= (\lambda - 1)\underbrace{\left(\frac{\lambda^3 - 7\lambda^2 + 14\lambda - 8}{\lambda - 1}\right)} \longrightarrow (*)$$

$$* \quad \frac{\lambda^3 - 7\lambda^2 + 14\lambda - 8}{\lambda - 1}$$, dividing leading coefficients of numerator and divisor

i-e $\quad \frac{\lambda^3}{\lambda} = \lambda^2 \Rightarrow$ Quotient $= \lambda^2$.

Now, Multiply $(d-1)$ by d^2

i-e:

$$d^2(d-1) = d^3 - d^2.$$

Now;

subtract, $(d^3 - d^2)$ from

$d^3 - 7d^2 + 14d - 8$

i-e:

$$(d^3 - 7d^2 + 14d - 8) - (d^3 - d^2)$$

$$= \cancel{d^3} - 7d^2 + 14d - 8 - \cancel{d^3} + d^2$$

$$\boxed{\text{Reminder} = -6d^2 + 14d - 8}$$

$$\therefore \frac{d^3 - 7d^2 + 14d - 8}{d-1} = d^2 + \left(\frac{-6d^2 + 14d - 8}{d-1}\right) \to (1)$$

Now;

$\frac{-6d^2 + 14d - 8}{d-1}$, divide the leading coefficients of numerator and the divisor,

i-e:

$$\frac{-6d^{\cancel{2}}}{\cancel{d}} = -6d \to \boxed{\text{Quotient} = -6d}$$

Now, multiply; $-6d$ by $d-1$

i-e;

$$(-6d)(d-1) = -6d^2 + 6d$$

Also: subtract $-6d^2 + 6d$ from

$-6d^2 + 14d - 8.$

i.e;

$$(-6\lambda^2+14\lambda-8)-(-6\lambda^2+6\lambda)$$

$$= \cancel{-6\lambda^2}+14\lambda-8+\cancel{6\lambda^2}-6\lambda$$

$$\boxed{\text{Reminder} = 8\lambda-8}$$

Therefore;

$$\frac{-6\lambda^2+14\lambda-8}{\lambda-1} = -6\lambda + \frac{8\lambda-8}{\lambda-1} \rightarrow (2)$$

So, Refine Eq 2) in Eq 1)

$$\therefore \text{Eq 1)} \Rightarrow \frac{\lambda^3-7\lambda^2+14\lambda-8}{\lambda-1} = \lambda^2+(-6\lambda)+\frac{8\cancel{\lambda}-8}{\cancel{\lambda-1}} \quad 8$$

$$\Rightarrow \frac{\lambda^3-7\lambda^2+14\lambda-8}{\lambda-1} = \lambda^2-6\lambda+8 \rightarrow (3)$$

Now; factorize " $\lambda^2-6\lambda+8$ "

i.e: $\lambda^2-6\lambda+8 = \lambda^2-2\lambda-4\lambda+8$

$$\Rightarrow \lambda^2-6\lambda+8 = \lambda(\lambda-2)-4(\lambda-2)$$

$$= (\lambda-2)(\lambda-4) \rightarrow (4)$$

★ -2λ
-4λ
-6λ

★ 2λ
$\times\, 4\lambda$
$+8\lambda^2$

Now, put Eq 4) in Eq ★

$$\therefore \text{Eq ★)} \Rightarrow (\lambda-1)\left(\frac{\lambda^3-7\lambda^2+14\lambda-8}{\lambda-1}\right) = (\lambda-1)(\lambda-2)(\lambda-4)$$

Now, put the above equation in Eq #)

∴ Eq #)

$$\Rightarrow -1(\lambda^3 - 7\lambda^2 + 14\lambda - 8) = -(\lambda-1)(\lambda-2)(\lambda-4) \rightarrow (4)$$

As;

Eq # :

$$-1(\lambda^2 - 7\lambda^2 + 14\lambda - 8) = 0$$

∴ Eq 4) $\Rightarrow$

$$-(\lambda-1)(\lambda-2)(\lambda-4) = 0 \longrightarrow (5)$$

Now,

using zero factor principle

i-e:

If $ab=0$, then $a=0$ OR $b=0$

or (both $a=0$ and $b=0$).

Therefore;

$\lambda - 1 = 0$, $\lambda - 2 = 0$, $\lambda - 4 = 0$

$\Rightarrow \boxed{\lambda = 1}$, $\Rightarrow \boxed{\lambda = 2}$, $\Rightarrow \boxed{\lambda = 4}$

So;

Eigenvalues are;

$\lambda = 1$, $\lambda = 2$, $\lambda = 4$.

Now;

Eigen vectors for : $\lambda = 1$:-

Solve for $(A - \lambda I)$:

i-e:

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \overset{\lambda=1}{(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \longrightarrow (a)$$

Now;

To solve; $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Reduce the matrix, Eq a)

i-e: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} a & \cdots & b \\ 0 & \ddots & \vdots \\ 0 & 0 & c \end{bmatrix}$ Reduce ↑ matrix to Row Echelon form;

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

* cancel leading coefficient in R_2 by Performing R_2 : $R_2 - (1) \cdot R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

* cancel leading coefficient in R_3 by Performing R_3 : $R_3 + (1) \cdot R_1$

$R_2 \longleftrightarrow R_3$

P.T.O

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$R_3 - \tfrac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

, cancel leading co-efficient in R_3 by performing R_3.

∴ Row_Echelon form of $(A-\lambda I)$ $= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Now; Reduce matrix to Reduced Row Echelon form, i-e:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftarrow \tfrac{1}{2}\cdot R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

cancel leading coefficient in R_1
by performing

$$R_1 \leftarrow R_1 - 1R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, the system associated with the Eigenvalue $\lambda = 1$.

$$(A - \lambda I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x + 0 + 0 \\ 0 + y + z \\ 0 + 0 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y + z \\ 0 \end{bmatrix},$$ This reduces to the following system of Equations,

$$\begin{cases} \boxed{x = 0} \\ y + z = 0 \Rightarrow \boxed{y = -z} \end{cases}$$, plus the values in $\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore \eta = \begin{bmatrix} 0 \\ -z \\ z \end{bmatrix} \quad ; \quad z \neq 0$$

So; let $z = 1$

$$\therefore \boxed{\Rightarrow \eta = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}} \text{ — (A)} \Rightarrow \text{Eigen vector of } \lambda = 1 \text{ of Matrix A.}$$

Also;

<u>Eigen vectors for $\lambda = 2$</u> :-

Solve: $(A - 2I) \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow (A - 2I) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow (A - 2I) = \begin{bmatrix} 2-2 & 1-0 & 1-0 \\ 1-0 & 3-2 & 2-0 \\ -1-0 & 1-0 & 2-2 \end{bmatrix}$$

$$\Rightarrow (A - 2I) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \longrightarrow (b)$$

Now, to solve : $(A-2I)\eta = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Reduce the Matrix $A-2I$,

i.e:

$$A-2I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

Reduce by Row-Echelon form;

$$\therefore \quad A-2I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$R_1 \longleftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

cancel leading co-efficient in R_3

by Performing : $R_3 \leftarrow R_3 + 1R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

P.T.O

(swap) $R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

cancel leading co-efficients in R_3 by performing: $R_3 \leftarrow R_3 - \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Row echelon form of } (A-2I).$$

Now;

Reduce the above matrix to Reduced Row Echelon form

i-e;

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftarrow \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

cancel leading co-efficient in Row R_1 by performing

$$R_1 \leftarrow R_1 - 1R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow$$ Reduced Row Echelon form.

Now, the system associated with the Eigen value $\lambda = 2$.

$$(A - 2I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (A - 2I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow (A - 2I)(\eta) = \begin{bmatrix} x + 0 + z \\ 0 + y + z \\ 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} x + z \\ y + z \\ 0 \end{bmatrix}$$

This Reduces to following system of Equations;

$$\begin{cases} x + z = 0 & \Rightarrow x = -z \\ y + z = 0 & \Rightarrow y = -z \end{cases}$$

Plug it into $\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\Rightarrow \quad \eta = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} , z \neq 0$$

$\therefore$ let ; $z = 1$

$$\Rightarrow \quad \eta = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \longrightarrow (B) :$$ Eigenvector of $\lambda = 2$ of Matrix A.

Similarly;

Eigenvectors for $\lambda = 4$:-

Solve: $$(A - \lambda I) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2-4 & 1-0 & 1-0 \\ 1-0 & 3-4 & 2-0 \\ -1-0 & 1-0 & 2-4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix} \longrightarrow (C)$$

5.5

Now, to solve, $(A-4I)\eta = 0$

we need to reduce matrix $(A-4I)$.

i-e,

$$A-4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Reduce matrix to Row Echelon form;

i-e,
$$\begin{bmatrix} a & & b \\ 0 & & \\ 0 & 0 & c \end{bmatrix}$$

$$\therefore \quad A-4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

cancel leading co-efficient in R_2 by performing $R_2 \leftarrow R_2 + \frac{1}{2} R_1$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1/2 & 5/2 \\ -1 & 1 & -2 \end{bmatrix}$$

cancel leading co-efficient in R_3 by performing $R_3 \leftarrow R_3 - \frac{1}{2} \cdot R_1$

P.T.O

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1/2 & 5/2 \\ 0 & 1/2 & -5/2 \end{bmatrix}$$

cancel leading co-efficient in row R_3 by performing $R_3 \leftarrow R_3 + 1R_2$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1/2 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$$ $\rightarrow$ Row Echelon form of $A - 4I$

Now, Reduce matrix to Reduced Row Echelon form; $\begin{bmatrix} 1 & - & b \\ 0 & \diagdown & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1/2 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftarrow -2 \cdot R_2$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

cancel leading co-efficient in row R_1 by performing $R_1 \leftarrow R_1 - 1R_2$

$$\sim \begin{bmatrix} -2 & 0 & 6 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow -\tfrac{1}{2} R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$ $\rightarrow$ Reduced Row Echelon form.

Now, The system associated with the Eigenvalues $\lambda = 4$.

$$(A-4I)\eta = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$; As; $(A-4I)\eta = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + 0 - 3z \\ 0 + 1y - 5z \\ 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This reduces to the following system of Equations;

$$\begin{cases} x - 3z = 0 & \Rightarrow \quad x = 3z. \\ y - 5z = 0 & \Rightarrow \quad y = 5z. \end{cases}$$

Plug these values in to

$$\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \eta = \begin{bmatrix} 3z \\ 5z \\ z \end{bmatrix} \qquad z \neq 0$$

$\therefore$ let; $z = 1$

$$\Rightarrow \eta = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}.$$

Hence; The Eigenvectors for $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$

are; $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$

or

Eigen vectors of: $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$

Result