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I.D NO 7614

SECTION B

SEMESTER 10th

SUBJECT Differential EQuatio

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Q NO 1: $x^3 y''' + 2x^2 y' + 2y - 10x + 10/x$

Ans: Solution:

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$$

$$x^3 D^3 y + 2x^2 D^2 + 2y = 10x + 10x^{-1}$$

$$(x^3 D^3 + 2x^2 D + 2)\, y = 10x + 10x^{-1} \quad \text{— eq ①}$$

let $x = e^t \Rightarrow t = \ln x$

$$xD = D$$

$$x^2 D^2 = D(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2)$$

Subtituting into eq ①

$$(\Delta - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta + 2)\, y = 10^x + 10x^{-1}$$

$$(\Delta^3 - \Delta^2 + 2)\, y = 10x + \frac{10x}{et}$$

Using Synthetic division

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$$\Delta^2 - 2\Delta + 2 = 0$$

Now using Quardatic formula

$a = 1, \quad b = -2, \quad c = 2$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a}$$

$$\Delta = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\Delta = \frac{2 \pm \sqrt{-4}}{2}$$

$$\Delta = \frac{2(1 \pm i)}{2}$$

$$\Delta = 1 \pm i$$

Since roots as complex

$$y_c = \bar{e}^{?}(c_1 \cos t + c_2 \sin t)$$

Now particular Integration :

$$y_p = \frac{1}{\Delta^3 - \Delta^2 + 2} \cdot 10e^{t} + \frac{1}{\Delta^3 - \Delta^2 + 2} \cdot 10/e^{t}$$

$$y_p = \frac{10e^t}{(1)^3 - (1)^2 + 2} + \frac{10e^t}{(1)^3 - (1)^2 + 2}$$

$$y_p = 5e^t + 5e^t$$

General Solution:

$$y = y_c + y_p$$

$$y = e^{-x}(c_1 \cos t + c_2 \sin t) + 5et + 5et$$

Put $e^t = x$ & $t = \ln x$

$$y = e^{-x}(c_1 \ln x + c_2 \sin \ln x) + 5e^{x} + 5e^{-x}.$$

Q NO 2 :-

$$x^3 \frac{d^3y}{dx} + 4x^2 \frac{d^2y}{dx} - 5x \frac{dy}{dx} - 15y = x^4$$

Solution: Let $\frac{d}{dx} = D$

$$x^3D^3y + 4x^2D^2y - 5xDy - 15y = x^4$$

$$x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$$

let $x = e^t \Rightarrow t = \ln x$

$$xD = D$$

$$x^2D^2 = D(D-1) = \Delta^2 - \Delta$$

$$x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2D$$

Now Substituting:

$$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$$

$$(\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2 - \Delta) - 5(\Delta) - 15)y = e^{4t}$$

$$\Delta^3 + \Delta^2 + 2\Delta - 13)y = e^{4t}$$

Synthetic division

$$\begin{array}{c|cccc}
5 & 1 & 1 & -7 & -15 \\
  &   & 3 & 12 & 15 \\
\hline
  & 1 & 4 & 5 & \boxed{0}
\end{array}$$

$$\Delta^2 + 4\Delta + 5 = 0$$

Quardatic formula:

$$\Delta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2-4(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm 2i}{2}$$

$$y_c = e^{3x}\left(c_1 \cos t + c_2 \sin t\right)$$

For $y_p = ?$

$$y_p = \frac{1}{\Delta^3 + \Delta^2 - 7\Delta - 15} e^{3t}$$

$$y_p = \frac{1}{(4)^3 + (4)^2 - 7(4) - 15} e^{4t}$$

$$y_p = \frac{1}{37} e^{4t}$$

Hence

$$y = y_c + y_p$$

$$y = c_1 \cos t + c_2 \sin t) + \frac{1}{37} e^{4t}$$

again put $t = \ln x$ & $x = \ln x$

$$y = e^{3x}\left(c_1 \cos \ln x + c_2 \sin \log x\right) + \frac{1}{37} e^{4x}.$$

Question #3:

$$x^2y'' + 2xy' - 6y = 10x^2$$

Solution:

$$y(1) = 1 \quad \& \quad y'(1) = -6$$

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y = 10x^2$$

$$\left(x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} - 6\right)y = 10x^2$$

put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$$x = e^t \quad \& \quad \log x = t$$

$$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$$

$$(\Delta^2 + \Delta - 6)y = 10e^{2t}$$

The characteristic Equation:

$$\Delta^2 + \Delta - 6 = 0$$

$$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$$

$$\Delta(\Delta+3) - 2(\Delta+3) = 0$$

$$\Delta + 3 = 0 \qquad \Delta - 2 = 0$$

$$\Delta = -3 \qquad \Delta = 2$$

Since roots are real & distinct

For

$$y_c = c_1e^{-3t} + c_2e^{2t}$$

For $y_p =$

$$y_p = \frac{1}{D^2 - D - 6} \cdot 10e^{2t}$$

$$= \frac{10}{D^2 - D - 6} e^{2t}$$

$$= 10 \frac{1}{0} e^{2t}$$

Now,

$$10 \frac{1}{\frac{d}{dD}(D^2 + D - 6)} e^{2t}$$

$$y_p = 2t e^{2t}$$

General Solution:

$$y = y_c + y_p$$

$$= c_1 e^{-3t} + c_2 e^{2t} + 2te^{2t}$$

$$y = c_1 x^{-3} + c_2 x^2 + 2(\log x) x^2 \quad \text{— (B)}$$

put $y(1) = 1$ i.e $x = 1$, $y = 1$ in (B)

$$1 = c_1 (1)^{-3} + c_2 (1)^2 + 2\log(1)$$

$$1 = c_1 + c_2 \longrightarrow (C)$$

Now differentiate eq (B) w.r.t x.

$$y' = -3c_1 x^{-4} + 2c_2 x + \frac{2}{x}(x^2) + 4x \log x$$

Now put $y'(1) = -6$ i.e $y' = -6$, $x = -6$

$$-6 = -3c_1 + 2c_2 + 2 + 0$$

$$\Rightarrow -6 - 2 = -3c_1 + 2c_2 + 2$$

$$\Rightarrow -8 = -3c_1 + 2c_2 \longrightarrow (D)$$

Now put $y'(1)$ with (3) & eq [illegible] (2)

$[illegible] = 4$

$[illegible] = [illegible]$

$[illegible] = 16$

$c_1 = 2$

$-8 = -3(2) + 2c_2$

$-8 = -6 + 2c_2$

$2c_2 = -8 + 6$

$c_2 = -1$.

Now put value of c_1 & c_2 in eq (B)

$y = 2x^{-2} - x^3 + 2\ln(x^2)$

$y = \frac{2}{x^2} - x^3 + 2x^2 \log x$.

Question # 04 :

$$x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = x^5$$

$y(1) = 4$, $y'(1) = 2$

Solution:

$$x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = x^5$$

$$\Rightarrow \left(x^2\frac{d^2}{dx^2} + 7x\frac{d}{dx} + 8\right)y = x^5 \text{ —— (A)}$$

[illegible] $= \Lambda \Rightarrow x^2 D^2 = \Lambda(\Lambda - 1) = \Lambda^2 - \Lambda$

$x = e^t \Rightarrow \log x = t$ in eq (A)

$(\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}$

$(\Delta^2 + 6\Delta + 5)\, y = e^{5t}$

By Quardatic formula:

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$D = -3 \pm 2$$

$y_c = c_1 e^{-5t} + c_2 e^{-t}$

For $y_p = ?$

$$y_p = \frac{1}{\Delta^2 + 6\Delta + 5} e^{5t}$$

$$y_p = \frac{1}{60} e^{5t}$$

Now General Solution is;

$y = y_c + y_p$

$y = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{60} e^{5t}$

$y = c_1 x^{-5} + c_2 x^{-1} + \frac{1}{60} x^5 \longrightarrow$ (B)

$x = 0$ put inthis eq.

Now in eq (B) $e^0 = 1$

put $y(0) = 2$ i.e $y = 2$ & $x = 2$

$2 = c_1 (2)^{-5} + c_2 (2)^{-1} + \frac{1}{60} (2)^5$

$$2 = -32c_1 - 2c_2 + \frac{1}{60}(32)$$

$$\frac{22}{15} = -32C_1 - 2C_2 \rightarrow ⓒ$$

Now diff. eq Ⓑ w.r. to "x"

$$y' = -5c_1, x^{-6} - c_2x^{-2} + \frac{1}{12}x^4$$

put $y'(1) = 2$ i.e $y' = 2$ & $x = 2$ in above eq.

$$2 = -5c_1 \cdot c_2^{-6} - c_2(2)^{-2} + \frac{1}{12}(2)^4$$

$$2 = -5c_1(-64) - c_2(4) + \frac{1}{12}(16)$$

$$2 = 320c_1 + 4c_2 + 4/3$$

$$2/3 = 320c_1 + 4c_2 \longrightarrow Ⓓ$$

$$\frac{-44}{15} = 64C_1 + 4C_2$$

$$c_1 = 580.$$

put value of c_1 in eq ⓒ

$$22/15 = -32(580) - 2c_2$$

$$\frac{18561}{-2} = c_2$$

$$-9280 = c_2.$$

Now put value of c_1 & c_2 in eq Ⓑ

$$y = 580 x^{-5} - 9280 x^{-1} + {}^{1}/_{60}\, x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60} x^5 .$$

Question # 05:

$$(x+1)^2 y'' - 3(x+1)' y + 4y = x^2$$

Solution:

$$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$(x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4)y = x^2$$

$$\Rightarrow [(x+1)^2 D^2 - 3(x+1) D + 4] y = x^2 \longrightarrow Ⓐ$$

put $(x+1)D = \Delta$ $\Rightarrow (x+1)^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x = e^t$ in eq Ⓐ

$$\Rightarrow [D^2 - \Delta - 3\Delta + 4] y = e^{2t}$$

$$(\Delta^2 - 4\Delta + 4) y = e^{2t}$$

$$(\Delta^2 - 4\Delta + 4) z = e^{2t}$$

For yc we find the roots

$$D^2 - 4D + 4 = 0$$

$$D^2 - 2\Delta - 2\Delta + 4 = 0$$

$$\Delta(\Delta - 2) - 2(\Delta - 2) = 0$$

$\Delta - 2 = 0$, $\Delta = 2$

$\Delta - 2 = 0$, $\Delta = 2$

so the roots are real & repeat.

The General Solution are;

$y = (c_1 + c_2 x)^{mx}$

$y = (c_1 + c_2 x)^{2x}$

For $y_p = ?$

$$y_p = \frac{1}{D^2 - 4D + 4}$$

$$y_p = \frac{2}{2\Delta - 4} e^{2t}$$

If we put "2"

$2D - 4 = 2(2) - 4 = 0$

we take again derivaties

$y_p = 7/2 \cdot e^{2t}$

$y = (c_1 + c_2 x)^{2t} + e^{2t}$.