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Q1 (a) Define differential equation along with 2 examples?

Ans = Differential Equation :—

"An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more Independent variables, is said to be a differential equation. (DE)".

Examples :—

1) $dy/dx = x$

2) $yy' + x = 0$

3) $dy/dx = 2x+3$

(b) Define a Separable differential equation (DE)?

Ans: Separable differential equation:
"A Separable differential equation can be expressed as the product of a function of $x$ and function of $y$.

$$dy/dx = g(x) \cdot h(y) \qquad h(y) \neq 0$$

(i) Solve the following Initial Value problem (IVP) using Separable DE and find the interval of valielity of the solution.

((a)) $y' = \dfrac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$

Ans: Solution :—

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

$$y^3\,dy = x(1+x^2)^{-1/2}\,dx$$

$$\int y^3\,dy = \int x(1+x^2)^{-1/2}\,dx$$

$$\frac{1}{-2y^2} = \sqrt{1+x^2} + C$$

$$-\frac{1}{2} = \sqrt{1} + C$$

$$C = \frac{-3}{2}$$

$$\frac{1}{-2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

$$y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Now finding Interval of validity :

$$3 - 2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4(1+x^2)$$

$$9/4 > 1+x^2$$

$$5/4 > x^2$$

$$-\sqrt{5}/2 < x < \sqrt{5}/2$$

$$x = 0.$$

Interval of validity.

Ans

(b) $y' = e^{-y}(2x-4) \qquad y(5) = 0$

Sol :–

Multiplying by $e^{y}$ and by $dx$.

$$e^{y}\,dy = (2x-4)\,dx.$$

$$\int e^{y}\,dy = \int (2x-4)\,dx$$

$$e^{y} = x^2 + 4x + C$$

(Natural log)

$$y = \ln(x^2 + 4x + C)$$

finding C :–

$$y(5) = \ln((5)^2 - 4(5) + C)$$

$$\ln(5 + C) = 0.$$

$$5 + C = 1$$

$$C = -4$$

$$y = \ln(x^2 - 4x - 4)$$

Ans

Q3:

(ii) $\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))\ y' = 0$

$y(5) = 0$

Sol:—

$M = \frac{2ty}{t^2+1} - 2t \qquad M_y = \frac{2t}{t^2+1}$

$N = \ln(t^2+1) - 2 \qquad N_t = \frac{2t}{t^2+1}$

Integrate the first one

$\psi(x,y) = \int \frac{2ty}{t^2+1} - 2t\, dy = y\ln(t^2+1)$

$- t + h(y)$

=> Now differentiate

$\psi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2$

$h'(y) = -2 \Rightarrow h(y) = -2y$

$$\psi(t,y) = y\ln(t^2+1) - t^2 - 2y$$

$$y\ln(t^2+1) - t^2 - 2y = C$$

$$C = -25$$

$$y(\ln(t^2+1) - 2 - t^2 = -25$$

$$y(t) = \frac{t^2 - 25}{\ln(t^2+1) - 2}$$

$$\ln(t^2+1) - 2 = 0$$

$$\ln(t^2+1) = 2$$

$$t^2 + 1 = e^2$$

$$\boxed{t = \pm\sqrt{e^2 - 1}}$$ Ans

$$\boxed{t = \pm\sqrt{e^2 - 1}}$$ Ans

Q3 Solve the following IVP for the exact equation and find the interval of validity for the solution :—

$\Rightarrow$ (i) $2xy - 9x^2 + (2y + x^2 + 1)\, dy/dx = 0,$

$y(0) = -3$

$\Rightarrow$ Solution :—

$M = 2xy - 9x^2 \quad , \quad M_y = 2x$

$N = 2y + x^2 \quad , \quad N_x = 2x$

Now, how do we find $\psi(x,y)$?

$\psi_x = M$

$\psi_y = N$

$\psi = \int M dx \quad$ or $\quad \psi = \int N dy$.

$\psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$

$h'(y) = 2y + 1$

$h(y) = \int 2(y) + 1\, dy = y^2 + y + k$

$\psi(x,y) = x^2 y - 3x^2 + y^2 + y + k$

$= y^2 + (x^2 + 1)y - 3 + k$

$$y^2+(x^2+1)y+3x^2+k=c$$

$$y^2+(x^2+1)\,y-3x^2=c-k$$

$$y^2+(x^2+1)\,y-3x^2=C$$

* Initial condition to find C

$$(-3)^2+(0+1)(-3)-3(0)^3=C$$

$$(9)+(-3)=C$$

$$C=6$$

Put the value of C

$$y^2+(x^2+1)y-3x^2-6=0$$

⇒ Using Quadratic formula

$$y(x)=\frac{-(x^2+1)\pm\sqrt{(x^2+1)^2-4(1)(-3x^2-6)}}{2(1)}$$

$$=\frac{-(x^2+1)\pm\sqrt{x^4+12x^3+2x^2+25}}{2}$$

$$-3=y(0)=\frac{-1\pm\sqrt{25}}{2}=\frac{-1\pm5}{2}=-3,2$$

$$y(x)=\frac{-(x^2+1)-\sqrt{x^4+12x^2+25}}{2}$$

$$x^4+12x^3+2x^2+25=0$$

Q2: (a) Solve the following IVP using linear differential ~~Equation~~ method.

(i) Explain the steps for Solving Linear differential Equation.

Ans: ⇒ Steps for Solving linear differential Equation :—

* Substitute $y = uv$ and
$$dy/dx = u\frac{dv}{dx} + v\, du/dx$$
into
$$dy/dx + P(x)y = Q(x)$$

* Factor the parts Involving v.
* Put v term equal to zero (this give a differential equation in u and x which can be Solved in next step)
* Solve using Separation of variables to find u.
* Substitue u back into the equation we got at step 2.

* Solve that to find v.
* Finally, Substitue u and v into $y = uv$ to get our solution!

(ii) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$

$y\left[\frac{\pi}{4}\right] = 3\sqrt{2}, \quad 0 \le x \le \frac{\pi}{2}$

Solution :-

$$y' + \frac{\sin(x)}{\cos(x)} y = 2\cos^2(x)\sin(x) - \frac{1}{\cos x}$$

$$y' + \tan(x)y = 2\cos^2(x)\sin x - \sec(x).$$

$$M(t) = e^{\int \tan(x)dx} = e^{\ln|\sec(x)|} = e^{\ln \sec(x)}$$

$$\sec(x)y^3 + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$\int \sec(x)\, y(x)'\, dx = \int 2\cos(x)\sin(x) - \sec^2(x)$$

$$\int (\sec(x)\, y(x))' \, dx = \int 2\cos(x)\sin(x) - \sec^2(x)$$

$$\sec(x)\, y(x) = \int \sin(2x) - \sec^2(x)\, dx$$

$$\sec(x)\, y(x) = -\tfrac{1}{2}\cos(2x) - \tan(x) + C$$

$$y(x) = -\tfrac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + C\cos(x)$$

$$= -\tfrac{1}{2}\cos(x)\cos(2x) - \sin(x) + C\cos(x)$$

Put the values of "y" & "x".

$$3\sqrt{2} = y(\pi/4) = -\tfrac{1}{2}\cos(\pi/4)\cos(\pi/2) - \sin(\pi/2)$$

$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + C\frac{\sqrt{2}}{2}$$

$$C = 7$$

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Ans