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Q1: $x^3y''' + 2x^2y'' + 2y = 10x + \frac{10}{x}$

Sol:-

$$x^3D^3y + 2x^2D^2y + 2y = 10x + \frac{10}{x}$$

$$(x^3D^3 + 2x^2D^2 + 2)y = 10x + \frac{10}{x} \quad \text{--- } A$$

let $x = e^t \Rightarrow t = \ln x$

$$xD = \Delta$$
$$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$$
$$x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$$

Substitute in eq A

$$(\Delta^3 - 3\Delta^2 + \cancel{2\Delta} + 2\Delta^2 - \cancel{2\Delta} + 2)y = 10x + 10x^{-1}$$

$$(\Delta^3 - \Delta^2 + 2)y = 10x + 10x^{-1}$$

Complementary Sol:- y_c :-

$$m^3 - m^2 + 2 = 0$$

-1	1	-1	0	2
		-1	2	-2
	1	-2	2	$\boxed{0}$

$$(\Delta+1)(\Delta^2 - 2\Delta + 2) = 0$$

$$\Delta^2 - 2\Delta + 2 = 0$$

Now using Quadratic formula.

$a = 1$, $b = -2$, $c = 2$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

②

$$\Delta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$\Delta = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{4}}{2} = \frac{2 + 2i}{2}$$

$$\Delta = 1 \pm 1i$$

Since root are real & also complex.

$$\cancel{y_c = c_1 e^{-1x} + e^{-x}(c_1 \cos t + c_2 \sin t)}$$

$$y_c = c_1 e^{-x} + e^{-x}(c_1 \cos t + c_2 \sin t)$$

Now particular Integration,

$$y_p = \frac{1}{\Delta^3 - \Delta^2 + 2} 10e^{t} + \frac{1}{\Delta^3 - \Delta^2 + 2} 10e^{-t}$$

$$y_p = \frac{1}{(1)^3 - (1)^2 + 2} 10e^{t} + \frac{1}{(-1)^3 - (-1)^2 + 2} 10e^{-t}$$

$$= \frac{\cancel{10}e^{t}}{\cancel{1} - \cancel{1} + \cancel{2}} + \frac{10e^{-t}}{-1-1+2=0} \quad 10e^{-t}$$

$$= \frac{5e^{t}}{} + \frac{10e^{-t}}{3\Delta^2 - 2\Delta}$$

$$y_p = 5e^{t} + \frac{10e^{-t}}{3(-1)^2 - 2(-1)}$$

$$= 5e^{t} + \frac{10e^{-t}}{3+2}$$

$$y_p = 5e^{t} + 5e^{-t}$$

General Solution:

$$y = y_p + y_c$$

$$y = c_1 e^{-x} + e^{-x}(c_1 \cos t + c_2 \sin t) + 5e^{t} + 5e^{-t}$$

replace $e^{t} = x$ & $t = \ln x$

$$\boxed{y = c_1 e^{-x} + e^{-x}(c_1 \cos \ln x + c_2 \sin \ln x) + 5x + 5x^{-1}}$$

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Q2 :- $x^3y''' + 4x^2y'' - 5xy' - 15y = x^4$

Solution :- $x^3D^3y + 4x^2D^2y - 5xDy - 15y = x^4$

$$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4 \quad \text{--- } A$$

Let $x = e^t$, $t = \ln x$

$xD = \Delta$

$x^2D^2 = \Delta^2 - \Delta$

$x^3D^3 = \Delta^3 - 3\Delta^2 + 2\Delta$

Substitute the values,

$$(\Delta^3 - 3\Delta^2 + 2\Delta + 4\Delta^2 - 4\Delta - 5\Delta - 15)y = x^4$$

$$(\Delta^3 + \Delta^2 - 7\Delta - 15)y = e^{4t}$$

For Complemtary Solution :- y_c :-

$$\Delta^3 + \Delta^2 - 7\Delta - 15 = 0$$

Synthetic Division :

$$\begin{array}{c|cccc} 3 & 1 & 1 & -7 & -15 \\ & & 3 & 12 & 15 \\ \hline & 1 & 4 & +5 & \underline{|0} \end{array}$$

$$(\Delta - 3)(\Delta^2 + 4\Delta + 5) = 0$$

$$\Delta^2 + 4\Delta + 5 = 0$$

using Quadratic formula,

$a = 1$, $b = +4$, $c = 5$

(5)

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\lambda = \frac{-4 \pm \sqrt{(4)^2-4(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} \Rightarrow \lambda = -2 \pm 1i$$

So $m_{1,2} = -2+i$, $m_{3,4} = -2-i$

$$y_c = e^{-2x}(c_1 \cos x + c_2 \sin x) + e^{-2x}(c_1 \cos(-x) + c_2 \sin(-x))$$

$$y_c = e^{-2x}(c_1 \cos x + c_2 \sin x) + e^{-2x}(c_1 \cos x - c_2 \sin x)$$

For $y_p = ?$

$$y_p = \frac{1}{\lambda^3+\lambda^2-7\lambda-15} e^{4t}$$

$$= \frac{1}{(4)^3+(4)^2-7(4)-15} e^{4t}$$

$$= \frac{1}{64+16-28-15} e^{4t}$$

$$y_p = \frac{e^{4t}}{37}$$

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$y = y_c + y_p$

$y = e^{-2x}(c_1\cos x + c_2\sin x) + e^{-2x}(c_1\cos x - c_2\sin x) + \frac{e^{4t}}{37}$

Again put $x = e^t$, $t = \ln x$

$\boxed{y = e^{-2x}(c_1\cos x + c_2\sin x) + e^{-2x}(c_1\cos x - c_2\sin x) + \frac{4x}{37}}$

Q3 :- $x^2y'' + 2xy' - 6y = 10x^2$; $y(1) = 1$
$y'(1) = -6$

Sol:- $x^2D^2y + 2xDy - 6y = 10x^2$

$(x^2D^2 + 2xD - 6)y = 10x^2$

let $x = e^t$, $t = \ln x$

$xD = \Delta$

$x^2D^2 = \Delta^2 - \Delta$

$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$

$(\Delta^2 + \Delta - 6)y = 10e^{2t}$

For complementary Equation: y_c :-

y_c = $\Delta^2 + \Delta - 6 = 0$

$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$

$\Delta(\Delta+3) - 2(\Delta+3) = 0$

$(\Delta+3)(\Delta-2) = 0$

$\Delta + 3 = 0 \quad , \quad \Delta - 2 = 0$

$\Delta = -3 \quad , \quad \Delta = +2$

Since roots are real & distinct

$$y_c = c_1 e^{-3t} + c_2 e^{2t}$$

for $y_p = ?$

$$y_p = \frac{1}{\Delta^2 + \Delta - 6} e^{2t}$$

$$= \frac{1}{(2)^2 + (2) - 6} = \frac{1}{4 + 2 - 6} = 0$$

$$y_p = \frac{1}{2\Delta + \Delta} e^{2t} = \frac{1}{2(2) + 2} e^{2t}$$

$$y_p = \frac{1}{4 + 2} e^{2t} \Rightarrow \frac{1}{6} e^{2t}$$

$$y_p = \frac{e^{2t}}{6}$$

So General Solution :-

$$y = y_c + y_p$$

$$y = c_1 e^{-3t} + c_2 e^{2t} + \frac{e^{2t}}{6}$$

put $x = e^t$, & $t = \ln x$

$$y = c_1 x^{-3} + c_2 x^2 + \frac{x^2}{6} \quad (*) \qquad y(1) = 1$$

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$y = c_1(1)^{-3} + c_2(1)^2 + \frac{(1)^2}{6}$

$y = c_1 + c_2 + \frac{1}{6}$

$(1) = c_1 + c_2 + \frac{1}{6}$

$c_1 + c_2 = 1 - \frac{1}{6}$

$= \frac{6-1}{6}$

$c_1 + c_2 = 5/6$ —— (1)

Now Differentiate eq (*) w.r.to x

$y' = -3c_1x^{-4} + 2c_2x + \frac{x}{3}$

Now put $y'(1) = -6$

$-6 = -3c_1(1)^{-4} + 2c_2(1) + \frac{(1)}{3}$

$-6 = -3c_1 + 2c_2 + \frac{1}{3}$

$-3c_1 + 2c_2 = -\frac{6}{1} - \frac{1}{3}$

$-3c_1 + 2c_2 = \frac{-18-1}{3} \Rightarrow -3c_1 + 2c_2 = \frac{-17}{3}$ (2)

After solving eq (1) & (2)

$\boxed{c_1 = 2}$ $\boxed{c_2 = -1}$

put in (*)

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$$\boxed{y = 2x^{-3} - x^2 + \frac{x^2}{6}} \longrightarrow \underline{\text{Ans}}$$

Q4:- $x^2y'' + 7xy' + 5y = x^5$; $y(0) = 2$ &
$y(1) = 2$

Solution #

$$x^2D^2y + 7xDy + 5y = x^5$$

$$(x^2D^2 + 7xD + 5)\,y = x^5$$

let $x = e^t$, $t = \ln x$

$$xD = \Delta$$

$$x^2D^2 = \Delta^2 - \Delta$$

$$(\Delta^2 - \Delta + 7\Delta + 5)\,y = x^5$$

$$(\Delta^2 + 6\Delta + 5)\,y = e^{5t}$$

for y_c :-

$$\Delta^2 + 6\Delta + 5 = 0$$

$$\Delta^2 + 5\Delta + \Delta + 5 = 0$$

$$\Delta(\Delta + 5) + 1(\Delta + 5) = 0$$

$$(\Delta + 5)(\Delta + 1) = 0$$

$$\Delta = -5 \quad , \quad \Delta = -1$$

Since roots are real & distinct,

$$y = c_1 e^{-5t} + c_2 e^{-t}$$

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For y_p :-

$$y_p = \frac{1}{D^2+6D+5} e^{5t}$$

$$= \frac{1}{(5)^2+6(5)+5} e^{5t}$$

$$= \frac{e^{5t}}{25+30+5} \Rightarrow y_p = \frac{e^{5t}}{60}$$

Now General Solution :-

$$y = y_c + y_p$$

$$y = c_1 e^{-5t} + c_2 e^{-t} + \frac{e^{5t}}{60} \quad \text{——} \quad A$$

$$y = c_1 x^{-5} + c_2 \cancel{x^{-1}} + x^5/60 \quad \text{——} \quad B$$

Now put $x = 2$ & $y = 2$ in eq B

$$\cancel{2 = c_1(0) + c_2(0) + (0)}$$

$$2 = c_1(2)^{-5} - c_2(2)^{-1} + \frac{1}{60}(2)^5$$

$$= -32 c_1 + (-2) c_2 + \frac{1}{60_{15}} (32)^{8}$$

$$-32c_1 - 2c_2 = \frac{32}{15} \quad \text{——} \quad C$$

Now Differentiate eq B w.r. to x

$$y' = -5c_1 x^{-6} - c_2 x^{-2} + \frac{5x^4}{60} \qquad y'(1) = 2$$

$$2 = -5c_1(1)^{-6} - c_2(1)^{-2} + \frac{5(1)^4}{60 \to 12}$$

$$2 = -5c_1 - c_2 + \frac{1}{12}$$

$$-5c_1 - c_2 = 2 - \frac{1}{12} \Rightarrow -5c_1 - c_2 = \frac{24-1}{12}$$

$$-5c_1 - c_2 = \frac{23}{12} \text{ --- } (D)$$

After solving (C) & (D) eqns:

$$\boxed{c_1 = 560} \quad \& \quad \boxed{c_2 = -9280}$$

putting these values in eq (B)

$$\boxed{y = 580x^{-5} - 9280x^{-1} + \frac{x^5}{60}} \quad \text{Ans}$$

Q5:-

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$$

Sol:

$$(x+1)^2 D^2 y - 3(x+1) D y + 4y = x^2$$

$$\left((x+1)^2 D^2 - 3(x+1)D + 4\right) y = x^2$$

let $x+1 = e^t \Rightarrow t = \ln(x+1)$

$$(x+1)D = \Delta$$

$$(x+1)^2 D^2 = \Delta^2 - \Delta$$

$$(\Delta^2 - \Delta - 3\Delta + 4) y = e^{2t}$$

$$(\Delta^2 - 4\Delta + 4) y = e^{2t}$$

for y_c :-

$$\Delta^2 - 4\Delta + 4 = 0$$

$$\Delta^2 - 2\Delta - 2\Delta + 4 = 0$$

$$\Delta(\Delta - 2) - 2(\Delta - 2) = 0$$

$$(\Delta - 2)(\Delta - 2) = 0$$

$$\Delta = 2, \quad \Delta = 2$$

Since roots are real & repeated

$$y_c = (c_1 + c_2) e^{2t}$$

For y_p :-

$$y_p = \frac{1}{D^2 - 4D + 4} e^{2t}$$

$(2)^2 - 4(2) + 4$

$4 - 8 + 4 = 0$

$$y_p = \frac{1}{2D - 4} e^{2t}$$

$$y_p = \frac{1}{2} e^{2t}$$ After taking 2 times derivative

$$y = y_c + y_p$$

$$y = (c_1 + c_2) e^{2t} + \frac{1}{2} e^{2t}$$

replace $t = \ln(x+1)$ & $x+1 = e^t$

$$y = (c_1 + c_2)(x+1)^2 + \frac{1}{2}(x+1)^2$$

$$y = \left\{ (c_1 + c_2) + \frac{1}{2} \right\} (x+1)^2$$

General solution

The END