Haris Ali Shah

7530

Sub-mitted to Madam Shumaila

Dated. 30/6/2020

Differential Equation.

Q1 Answer to Q1 (a)

Given

$$\frac{d^2w}{dt^2} = c^2 \frac{d^2w}{dx^2}$$

Now

$$\frac{dw}{dt} = \frac{d}{dt}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{d}{dt}(\sin(x+ct)) + \frac{d}{dt}(\cos(2x+2ct))]$$

$$\frac{dw}{dt} = c\cos(x+ct) - 2c\sin(2x+2ct)$$

$$\frac{d^2w}{dt^2} = \frac{d}{dt}\left[c\cos(x+ct) - 2c\sin(2x+2ct\right]$$

$$\frac{d^2w}{dt^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)$$

Scanned with CamScanner

Now

$$\frac{dw}{dx} = \frac{d}{dx}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{dw}{dx} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{d^2w}{dx^2} = \frac{d}{dx}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{d^2w}{dx^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$\Rightarrow -c^2\sin(x+ct) - 4c^2\cos(2x+2ct) = c^2\left[-\sin(x+ct) - 4\cos(2x+2ct)\right]$$

$$-c^2\cancel{\sin}(x+ct) - 4c^2\cancel{\cos}(2x+2ct) = -c^2\cancel{\sin}(x+ct) - 4c^2\cancel{\cos}(2x+2ct)$$

$$0 = 0 \quad \text{(satisfied)}$$

Q1 (b)

$w = \tan(2x + ct)$

Now

$$\frac{dw}{\partial t} = c\sec^2(2x + ct)$$

$$\frac{d^2w}{dt^2} = \frac{d}{\partial t}\left(c\sec^2(2x + ct\right)$$

$$= c^2 \cdot 2\sec^2(2x+ct)\tan(2x+ct)$$

Now

$$\frac{dw}{\partial x} = 2\sec^2(2x+ct)$$

$$\frac{\partial^2 w}{dt^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

$$\Rightarrow 4c^2 \cancel{\sec^3}(2x+ct)\tan(2x+ct) = 4c^2\cancel{\sec^3}(2x+ct)\tan(2x+ct)$$

$0 = 0$ (satisfied).

Q2 Answer to Q 2

A2 Given function is

$$F(x) = \begin{cases} x; & -\pi < x \le 0 \\ 2x; & 0 \le x \le \pi \end{cases}$$

we have to find the fourier Co-efficients a_0, a_n & b_n

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[0 - \frac{\pi^2}{2}\right] + \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^2}{\cancel{2}} - 0\right]$$

$$a_0 = \frac{-\pi}{2} + \pi = \frac{\pi}{2} \rightarrow (a)$$

Now a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (x \cos nx\, dx) + \frac{1}{\pi}\int_{0}^{\pi} (2x \cos nx)\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} +$$

$$\left[\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}\right)$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n+2(-1)^n-2}{n^2}\right] = \frac{(-1)^n-1}{\pi n^2}$$

So

$$a_n = \left\{\begin{array}{l} \frac{-2}{\pi n^2} \; ; \text{ if } n \text{ is odd} \\ 0 \; ; \text{ if } n \text{ is even} \end{array}\right\} \rightarrow (b)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx \, dx = \frac{1}{\pi}\int_{-\pi}^{0} x \sin nx \, dx + \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0} +$$

$$\frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi \cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-\pi \cos n\pi}{n}\right]$$

$$b_n = \frac{-3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}$$

So the required fourier is

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Q3 Answer to Q3

A3 Given

$$y'' - 4y' + 13y = 8\sin 3x$$

we have to find $y = y_c + y_p$

For y_c (Auxilary Eqn) Eqn is;

$$m^2 - 4m + 13 = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16-52}}{2} \Rightarrow m = \frac{4+6i}{2}$$

$$m = 2 \pm 3i \quad , \quad \alpha = 2 \text{ \& } \beta = 3$$

So $y_c = e^{2x}\{C_1\cos 3x + C_2 \sin 3x\}$

For y_p let

$$y_p = \text{Img} = \left(\frac{1}{m^2 - 4m + 13} 8e^{3ix} \right)$$

$$= 8\, \mathrm{Imag}\; \frac{e^{3ix}}{(3i)^2 - 4(3i) + 13}$$

$$= 8\, \mathrm{Imag}\; \frac{e^{3ix}}{-9 - 12i + 13}$$

$$= 8\, \mathrm{Imag}\; \frac{e^{3ix}}{4 - 12i}$$

$$y_p = 2\, \mathrm{Imag}\; \frac{e^{3ix}}{(1-3i)} \times \frac{(1+3i)}{(1+3i)}$$

$$y_p = 2\, \mathrm{Imag}\; \frac{(1+3i)(e^{3ix})}{1^2 - (3i)^2}$$

$$y_p = 2\, \mathrm{Imag}\; \frac{(1+3i)(e^{3ix})}{10}$$

$$y_p = \frac{2}{10}\left(\mathrm{Imag}\, (1+3i)(\cos 3x + i \sin 3x)\right)$$

$$y_p = \frac{2}{10}(\sin 3x + 3\cos 3x)$$

So

General solution is

$$y = y_c + y_p$$

$$y = C_1 e^{2x}\cos 3x + C_2 e^{2x}\sin 3x + \frac{2}{10}(\sin 3x + 3\cos 3x$$

Now using the initial condition
$y(0) = 1$

$$y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 + \frac{2}{10}(\sin 0 + 3\cos(0))$$

$$1 = C_1(1) + 0 + 0 + \frac{2}{10}(3(1))$$

$$1 = C_1 + \frac{6}{10}$$

$$C_1 = 1 - \frac{6}{10}$$

$$C_1 = 4/10 = 2/5$$

Scanned with CamScanner

Now Again using the initial condition

$y'(0) = 2$

So,

$$y' = C_1 2e^{2x}\cos 3x + C_1 e^{2x}(-3\sin 3x) + C_2 2e^{2x}\sin 3x + C_2 e^{2x}(3\cos 3x) + \frac{2}{10}(\cos 3x - 3\sin 3x$$

$$y'(0) = C_1 2e^{0}\cos 0 + C_1 e^{0}(-3\sin(0)) + C_2 2e^{0}\sin 0 + C_2 e^{0}(3\cos(0)) + \frac{2}{10}(\cos(0) - 3(\sin(0))$$

$$2 = 2C_1 + 0 + 0 + C_2 3(1) + \frac{2}{10}(1 - 3(0))$$

$$2 = 2C_1 + 3C_2 + \frac{2}{10}$$

using $C_1 = 2/5$

$$2 = 2\left(\frac{2}{5}\right) + 3C_2 + \frac{2}{10}$$

Scanned with CamScanner

$$\Rightarrow \frac{1}{3}\left\{2-\frac{4}{5}-\frac{2}{10}\right\} = c_2$$

$$c_2 = \frac{1}{3}\left(\frac{20-8-2}{10}\right) = 1/3$$

So the General solution is

$$y = \frac{2}{5}e^{2x}\cos 3x + \frac{1}{3}e^{2x}\sin 3x + \frac{2}{10} x \boxed{\sin 3x + 3\cos 3x}.$$

The required solution.

Answer to Q.4

(A4) Sol

Already in symbolic form

$(D^2 - DD')z = \cos x \cos 2y \rightarrow (a)$

put A.E $D^2 - DD' = 0$

we know that

$\frac{D}{D'} = m$ i-e $D = m$; $D' = 1$

$\Rightarrow m^2 - m = 0$

$m = 0, 1$

Now $C.F = f_1(y) + f_2(y + x)$

From eq (a)

$P.I = \frac{1}{D^2 - D.D'} \cos x \cos 2y$

As

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$C.F = f_1(y-x) + x f_2(y-x)$$

$$PI = \frac{1}{D^2+2DD'+D'^2}\left[2(y-x) + \sin(x-y)\right]$$

$$= \frac{1}{(D+D')^2}\left[2(y-x) + \sin(x-y)\right]$$

By General method,

$m = -1;\ y - x = c$

$$= \frac{1}{D+D'}\int\left[2c + \sin(-c)\right]dx$$

$$= \frac{1}{D+D'}\left[2cx - \sin(c)\,x\right]$$

Now Replacing c by $y-x$

$$= \frac{1}{D+D'}\left[2x(y-x) - x\sin(y-x)\right]$$

Again putting $y - x = c$

$$= \int (2xc - x\sin c)\,dx \Rightarrow cx^2 - \frac{x^2}{2}\sin c$$

Replacing c by $y-x$

$$= x^2(y-x) - \frac{x^2}{2}\sin(y-x) = x^2y - x^3 + \frac{x^2}{2}\sin(x-y)$$

Hence the required solution

$$Z = C.F + P.I = f_1(y-x) + x f_2(y-x) + x^2y - x^3 + 1/2\, x^2 \sin(x-y))$$

Scanned with CamScanner