Assignment

ID: 11533

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Semester: 12th

Subject: Differential Equation

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Q1 Use any of the method for solving the ordinary differential equations as given below.

Solve and graph the solution. Show the details of your work.

12 $x^2y'' - 4xy' + 6y = 0$, $y(1) = 0.4$
$y'(1) = 0$.

Solution:

Lets Substitute:

$$y = x^m,\ y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$$

into the given ODE. This gives:

$$x^2m(m-1)x^{m-2} - 4xmx^{m-1} + 6x^m = 0$$

$$\cancel{x^2}m(m-1)x^m \cdot \cancel{x^{-2}} - 4\cancel{x}mx^m\cancel{x^{-1}} + 6x^m = 0$$

We can see that x^m is a common factor, dropping it gives:

$$m(m-1) - 4m + 6 = 0 \Leftrightarrow m^2 - 5m + 6 = 0$$

So, $y = x^m$ is a solution of the given ODE if m is a root of the equation

Let's find the roots of the equation.

$$m^2 - 5m + 6 = 0 \Leftrightarrow m_{\frac{1}{2}} = \frac{5 \pm \sqrt{(-5)^2 + 4\cdot 6}}{2}$$

$$m_{\frac{1}{2}} = \frac{5 \pm 1}{2}$$

So, it has the ditinct real roots:

$$m_1 = 3 \wedge m_2 = 2$$

Real different ~~modes~~ roots m_1 and m_2 provide two real solutions:

$$y_1 = x^{m_1} = x^3 \wedge y_2 = x^{m_2} = x^2$$

Their quotient is not constant

So the solution y_1 and y_2 are linearly independent and constitute a basis of solution for the given ODE, for all x for which $y_1, y_2 \in \mathbb{R}$

So the general solution is:

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 x^3 + c_2 x^2$$
$$\Rightarrow y' = 3c_1 x^2 + 2c_2 x$$

Now determining c_1 and c_2 from IVP:

$$\begin{cases} 0.4 = y(1) = c_1 \cdot 1^3 + c_2 \cdot 1^2 \\ 0 = y'(1) = 3c_1 \cdot 1^2 + 2c_2 \cdot 1 \end{cases} \Rightarrow \begin{cases} 0.4 = c_1 + c_2 \\ 0 = 3c_1 + 2c_2 \end{cases}$$

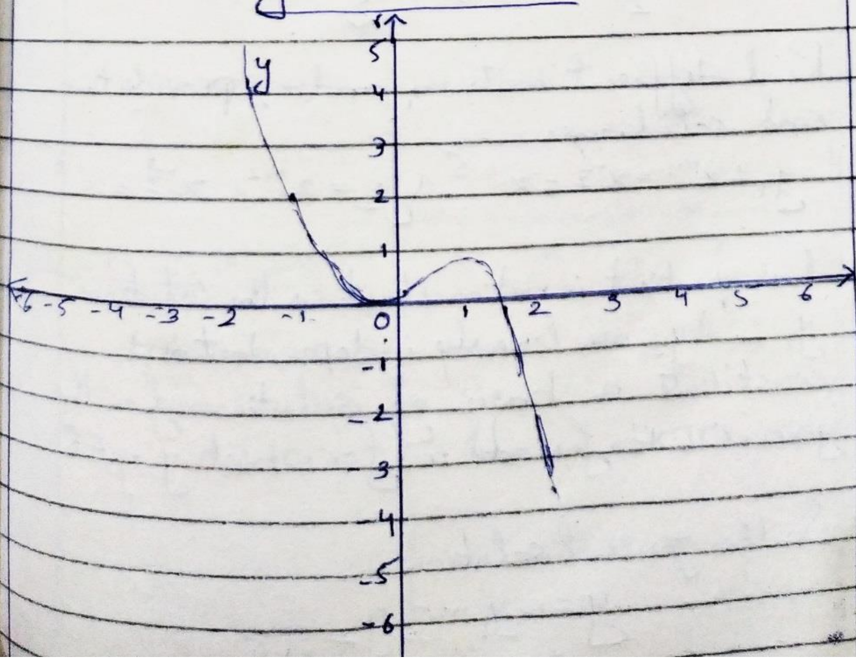
$$\Rightarrow \begin{cases} 0.4 - c_2 = c_1 \\ 0 = 3(0.4 - c_2) + 2c_2 \end{cases}$$

$$\Rightarrow \begin{cases} 0.4 - c_2 = c_1 \\ 0 = 1.2 - c_2 \end{cases} \Rightarrow \begin{cases} 0.4 - c_2 = c_1 \\ 1.2 = c_2 \end{cases}$$

$$\Rightarrow \begin{cases} 0.4 - 1.2 = c_1 \\ 1.2 = c_2 \end{cases} \Rightarrow \begin{cases} -0.8 = c_1 \\ 1.2 = c_2 \end{cases}$$

The particular solution of the IVP is:

$$y = -0.8x^3 + 1.2x^2$$



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13 $x^2y'' + 3xy' + 0.75y = 0$, $y(1) = 1$, $y'(1) = -1.5$

Solution:

Let's substitute:

$$y = x^m,\ y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$$

into the given ODE. That gives:

$$x^2 m(m-1)x^{m-2} + 3x\,m x^{m-1} + 0.75x^m = 0$$

$$\cancel{x^2} m(m-1)x^m \cdot \cancel{x^{-2}} + 3\cancel{x} m x^m \cancel{x^{-1}} + 0.75x^m = 0$$

We can see that x^m is a common factor, dropping it gives:

$$m(m-1) + 3m + 0.75 = 0 \Leftrightarrow m^2 + 2m + 0.75 = 0$$

So, $y = x^m$ is a solution of the given ODE if m is a root of the equation

Let's find the roots of the equation

$$m^2 + 2m + 0.75 = 0 \Leftrightarrow m_{\frac{1}{2}} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 0.75}}{2}$$

$$m_{\frac{1}{2}} = \frac{-2 \pm 1}{2}$$

So, it has the distinct real roots

$$m_1 = -\frac{1}{2} \wedge m_2 = -\frac{3}{2}$$

Real different roots m_1 and m_2 provide two real solutions:

$$y_1 = x^{m_1} = x^{-\frac{1}{2}} = x^{-0.5} \wedge y_2 = x^{m_2} = x^{-\frac{3}{2}} = x^{-1.5}$$

Their quotient is not constant, so the solutions y_1 and y_2 are linearly independent and constitute a basis of solutions for the given ODE, for all x for which $y_1, y_2 \in \mathbb{R}$

So the general solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1x^{-0.5} + c_2x^{-1.5}$$

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$y' = 0.5c_1 x^{-1.5} - 1.5c_2 x^{-2.5}$

Now all we need to determine is c_1 and c_2 from IVP:

$$\begin{cases} 1 = y(1) = c_1 \cdot 1^{-0.5} + c_2 \cdot 1^{-1.5} \\ -1.5 = y'(1) = -0.5c_1 \cdot 1^{-1.5} - 1.5c_2 \cdot 1^{-2.5} \end{cases}$$

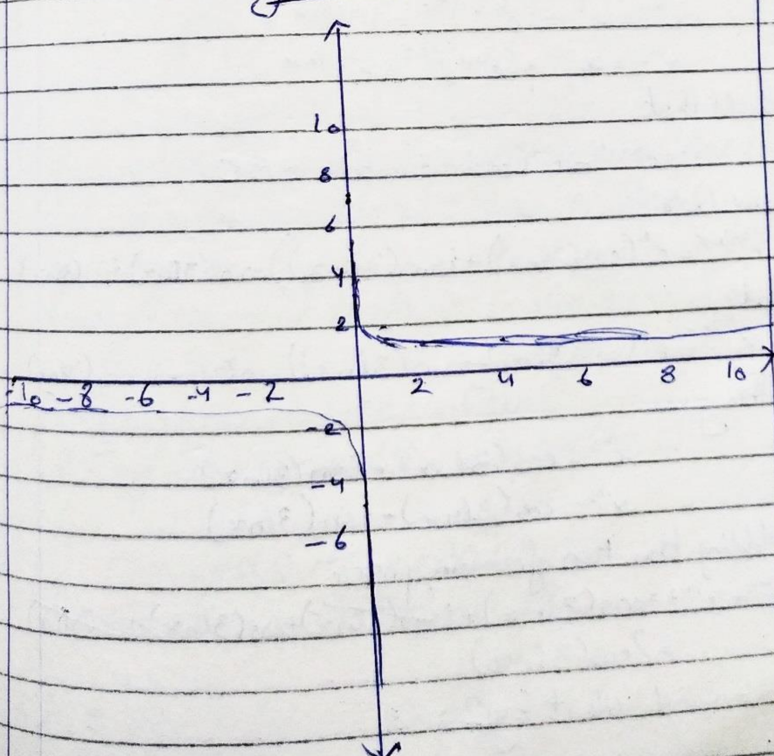
$$\Rightarrow \begin{cases} 1 = c_1 + c_2 \\ -1.5 = -0.5c_1 - 1.5c_2 \ //(:-0.5) \end{cases} \Rightarrow \begin{cases} 1 = c_1 + c_2 \\ 3 = c_1 + 3c_2 \end{cases}$$

$$\Rightarrow \begin{cases} 1 - c_2 = c_1 \\ 3 = 1 - c_2 + 3c_2 \end{cases} \Rightarrow \begin{cases} 1 - c_2 = c_1 \\ 2 = 2c_2 \ //(:2) \end{cases}$$

$$\Rightarrow \begin{cases} 1 - c_2 = c_1 \\ 1 = c_2 \end{cases} \Rightarrow \begin{cases} 0 = c_1 \\ 1 = c_2 \end{cases}$$

The particular solution for IVP is

$$\underline{y = x^{-1.5}}$$



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14 $x^2y'' + xy' + 9y = 0$, $y(1) = 0$, $y'(1) = 2.5$

Solution:

Let's substitute:

$$y = x^m, \quad y'' = m(m-1)x^{m-2}$$

into the given ODE. This gives:

$$x^2 m(m-1)x^{m-2} + mx^m + 9x^m = 0$$

$$\cancel{x^2} m(m-1)x^m \cdot \cancel{x^{-2}} + mx^m + 9x^m = 0$$

We can see that x^m is a common factor, dropping it gives:

$m(m-1) + m + 9 = 0 \Leftrightarrow m^2 - \cancel{m} + \cancel{m} + 9 = 0 \Leftrightarrow m^2 + 9 = 0$

So, $y = x^m$ is a solution of the given ODE if m is a root of the equation.

Let's find the roots of the equation.

$m^2 + 9 = 0 \Leftrightarrow m^2 - (2i)^2 = 0 \Leftrightarrow (m - 3i)(m + 3i) = 0$.

So, it has the complex conjugate roots:

$$m_1 = 3i \wedge m_2 = -3i$$

Now, we use the fact that $x = e^{\ln x}$:

$$x^{m_1} = x^{3i} = (e^{\ln x})^{3i} = e^{3i\ln x}$$

$$x^{m_2} = x^{-3i} = (e^{\ln x})^{-3i} = e^{-3i\ln x}$$

Recall that

$$e^z = e^{a+ib} = e^a(\cos b + i\sin b), \quad z \in \mathbb{C}$$

so we have

$$e^{3i\ln x} = e^0(\cos(3\ln x) + i\sin(3\ln x)) = \cos(3\ln x) + i\sin(3\ln x)$$

and

$$e^{-3i\ln x} = e^0(\cos(3\ln x) - i\sin(3\ln x)) = \cos(3\ln x) - i\sin(3\ln x)$$

This gives:

$$x^{m_1} = \cos(3\ln x) + i\sin(3\ln x)$$

$$x^{m_2} = \cos(3\ln x) - i\sin(3\ln x)$$

Adding this two formula gives:

$$x^{m_1} + x^{m_2} = \cos(3\ln x) + \cancel{i\sin(3\ln x)} + \cos(3\ln x) - \cancel{i\sin(3\ln x)}$$

$$= 2\cos(3\ln x)$$

Now, divide it by 2:

$$\frac{x^{m_1}+x^{m_2}}{2} = \frac{\cancel{2}\cos(3\ln x)}{\cancel{2}} = \cos(3\ln x)$$

Next, subtract the second formula from the first, and divide it by to $2i$ after that

$$x^{m_1}-x^{m_2} = \cos(3\ln x) + i\sin(3\ln x) - \cos(3\ln x) \cancel{+i}$$
$$+ i\sin(3\ln x))$$
$$= 2i\sin(3\ln x)$$

Divide it by $2i$:

$$\frac{x^{m_1}-x^{m_2}}{2i} = \frac{\cancel{2i}\sin(3\ln x)}{\cancel{2i}} = \sin(3\ln x)$$

By the superposition principle, $\cos(3\ln x)$ and $\sin(3\ln x)$ are the solutions of the Euler-Cauchy equation:

Their quotient is not constant, so the solutions

$$y_1 = \cos(3\ln x) \wedge y_2 = \sin(3\ln x)$$

are linearly independant and form a bases of solutions. So the general solution is:

$$y = c_1y_1 + c_2y_2$$

$$= c_1\cos(3\ln x) + c_2\sin(3\ln x)$$

$$\Rightarrow y' = -c_1\sin(3\ln x)\cdot(3\ln x)' + c_2\cos(3\ln x)\cdot(3\ln x)'$$

$$= -\frac{3c_1}{x}\sin(3\ln x) + \frac{3c_2}{x}\cos(3\ln x)$$

Now, all we need to do is to determine c_1 and c_2 from IVP:

$$\begin{cases} 0 = y(1) = c_1\cos(3\ln 1) + c_2\sin(3\ln 1) \\ 2.5 = y'(1) = -3c_1\sin(3\ln 1) + 3c_2\cos(3\ln 1) \end{cases}$$

$$\Rightarrow \begin{cases} 0 = c_1\cos(0) + c_2\sin(0) \\ 2.5 = -3c_1\sin(0) + 3c_2\cos(0) \end{cases}$$

$$\Rightarrow \begin{cases} 0 = c_1 \\ \frac{5}{2} = 3c_2 \quad /:3 \end{cases} \Rightarrow \begin{cases} 0 = c_1 \\ \frac{5}{6} = c_2 \end{cases}$$

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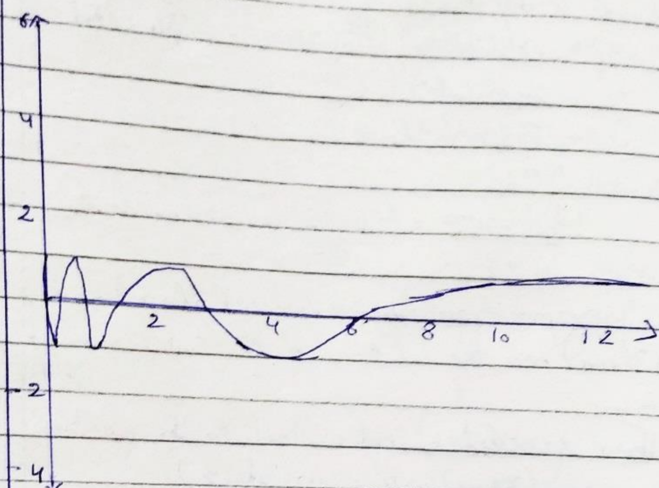
The partial solution of the IVP is

$$y = \frac{5}{6}\sin(3\ln(x))$$



15 $x^2y'' + 3xy' + y = 0$, $y(1) = 3.6$, $y'(1) = 0.4$

Solution:

Let's substitute:

$$y = x^m,\ y'' = m(m-1)x^{m-2}$$

into the given ODE. This gives:

$$x^2 m(m-1)x^{m-2} + 3mx^m + x^m = 0$$

$$x^2 m(m-1)x^m \cdot x^{-2} + 3mx^m + x = 0$$

We can see that x^m is a common factor, dropping

$m(m-1) + 3m + 1 = 0 \Leftrightarrow m^2 - m + 3m + 1 = 0 \Leftrightarrow m^2 + 2m + 1 = 0$

So, $y = x^m$ is a solution of the given ODE if m is a root of the equation

Let's find the roots of the equation

$m^2 + 2m + 1 = 0 \Leftrightarrow (m+1)^2 = 0$

So, it has the real double root.

$$m = 1$$

Real double root m provides a real solution

$y_1 = x^m = x^{-1} = \frac{1}{x}$

We can find a second linearly independent solution y_2 using the method of reduction of order.

First, we need to write the given ODE in the standard form.

$$y'' + \frac{3}{x} y' + \frac{1}{x^2} y = 0$$

Now we see that

$$p(x) = 3 \cdot \frac{1}{x} \Rightarrow \int p\,dx = 3\ln|x|$$

put $y_2 = uy_1$ in here

where $u = \int U\,dx$ & $U = \frac{1}{y_1^2} e^{-\int p\,dx}$

Let's find U:

$$e^{-\int p\,dx} = e^{-3\ln|x|} = \left(e^{\ln|x|}\right)^{-3} = x^{-3}$$

$$\Rightarrow U = x^{-3} \cdot \frac{1}{x^2} = x^{-3+2} = x^{-1} = \frac{1}{x}$$

By integration, we have:

$$u = \int \frac{dx}{x} = \ln|x|$$

So

$$y_2 = uy_1 = y_1 \ln x = \frac{1}{x} \cdot \ln x$$

Since their quotient is not constant, y_1 and y_2 are linearly independent and constitute a basis of solutions for the given ODE, for all x for which $y_1, y_2 \in R$.

So the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \frac{1}{2} + c_2 \frac{1}{2} \ln x$$

$$= \frac{1}{x}(c_1 + c_2 \ln x)$$

$\Rightarrow$ Product rule:

$$y' = (x^{-1})'(c_1 + c_2 \ln x) + x^{-1}(c_1 + c_2 \ln x)'$$
$$= -x^{-2}(c_1 + c_2 \ln x) + \frac{1}{x} c_2 \cdot \frac{1}{x}$$
$$= \frac{1}{x^2}(-c_1 - c_2 \ln x + c_2)$$

Now: determining c_1 and c_2 from IVP.

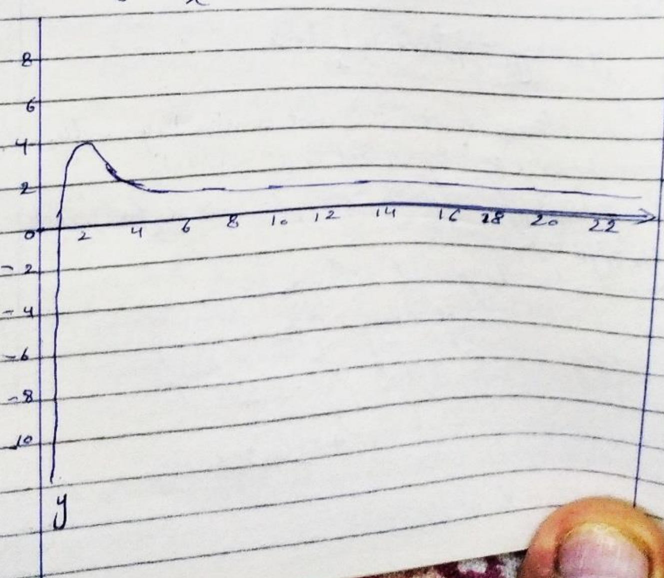
$$\begin{cases} 3.6 = y(1) = \frac{1}{1}(c_1 + c_2 \ln 1) \\ 0.4 = y'(1) = \frac{1}{1^2}(-c_1 - c_2 \ln 1 + c_2) \end{cases}$$

$$\begin{cases} 3.6 = c_1 \\ 0.4 = -c_1 + c_2 \end{cases} \rightarrow \begin{cases} 3.6 = c_1 \\ 0.4 = -3.6 + c_2 \end{cases}$$

$$\Rightarrow \begin{cases} 3.6 = c_1 \\ 4.0 = c_2 \end{cases}$$

The particular solution of IVP is

$$y = \frac{1}{x}\left[3.6 + 4.0(x)\right]$$



$(x^2D^2-3xD+4I)y=0,\ y(1)=-\pi,\ y'(1)=2\pi$

Solution

First we need to apply given operator to the given function

$$x^2D^2y-3xDy+4Iy=x^2D(Dy)-3xDy+4y$$
$$=x^2y''-3xy'+4y$$

Let's solve the equation.

$$x^2y''-3xy'+4y=0.$$

Let's substitute:

$$y=x^m,\ y'=mx^{m-1},\ y''=m(m-1)x^{m-2}$$

into the given ODE. This gives:

$$x^2m(m-1)x^{m-2}-3xmx^{m-1}+4x^m=0$$
$$\cancel{x^2}m(m-1)x^m\cancel{x^{-2}}-3xmx^m\cancel{x^{-1}}+4x^m=0$$

We can see that x^m is a common factor, dropping it gives

$$m(m-1)-3m+4=0 \Leftrightarrow m^2-4m+4=0$$

So, $y=x^m$ is a solution of the given ODE if m is a root of the equation

Let's find the root of equation:

$$m^2-4m+4=0 \Leftrightarrow (m-2)^2=0$$

So, it has the real double roots

$$m=2$$

~~Root~~ Real double root m provides a real solution

$$y=x^m=x^2$$

~~We~~ ~~can~~ We can find a second linearly independent solution y_2 using the method of reduction of order:

First, we need to write the given ODE in the standard form:

$$y''-\frac{3}{x}\cdot y'+\frac{4}{x^2}\cdot y=0$$

Now: $p(x)=-3\cdot\frac{1}{x} \Rightarrow \int p\,dx=-3\ln|x|$

Put

where: $y_2 = vy_1$

$$v = \int U\,dx \wedge U = \frac{1}{y_1^2} e^{-\int p\,dx}$$

Let's find U

$$e^{-\int p\,dx} = e^{3\ln|x|} = (e^{\ln|x|})^3 = x^3$$

$$\Rightarrow U = x^3 \cdot \frac{1}{(x^2)^2} = x^{3-4} = x^{-1} = \frac{1}{x}$$

By integration, we have

$$v = \int \frac{dx}{x} = \ln|x|$$

So

$$y_2 = vy_1 = y_1 \ln x = x^2 \ln x.$$

Since theirs quotient is not constant y_2 and y_1 are linearly independant and constitute a basis of solution for the given ODE, for all x for which $y_1, y_2 \in R$.

So the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 x^2 + x^2 \ln x$$

$$= x^2 (c_1 + c_2 \ln x)$$

$$y' = (x^2)'(c_1 + c_2 \ln x) + x^2 (c_1 + c_2 \ln x)'$$

$$= 2x(c_1 + c_2 \ln x) + c_2 x^2 \cdot \frac{1}{x}$$

$$= 2c_1 x + 2c_2 x \ln x + c_2 x$$

$$= 2c_1 x + c_2 x (2\ln x + 1)$$

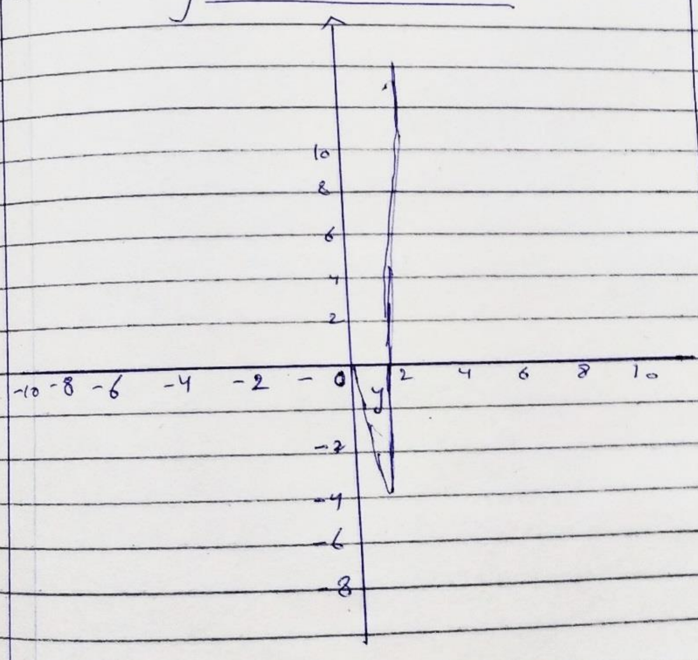
Now determining c_1 and c_2 from IVP.

$$\begin{cases} -\pi = y(1) = 1^2(c_1 + c_2 \ln 1) \\ 2\pi = y'(1) = 2c_1 + c_2(2\ln 1 + 1) \end{cases}$$

$$\begin{cases} -\pi = c_1 \\ 2\pi = 2c_1 + c_2 \end{cases} \Rightarrow \begin{cases} -\pi = c_1 \\ 4\pi = c_2 \end{cases}$$

Their particular solution of the IVP is

$$\underline{y = x^2(-\pi + 4\pi \ln x)}$$



$$\underline{y = x^2\pi(4\ln x - 1)}$$

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17 $(x^2D^2+xD+I)y=0$, $y(1)=1$, $y'(1)=1$

Solution:

First we need to apply the given operator to the given function:

$$x^2D^2y+xDy+Iy=x^2D(Dy)+xDy+y$$

$$=x^2y''+xy'+y$$

$$x^2y''+xy'+y=0$$

Let's substitute:

$$y=x^m, y'=mx^{m-1}, y''=m(m-1)x^{m-2}$$

That gives:

$$x^2m(m-1)x^{m-2}+xmx^{m-1}+x^m=0$$

$$\cancel{x^2}m(m-1)x^m\cdot \cancel{x^{-2}}+\cancel{x}mx^m\cancel{x^{-1}}+x^m=0$$

We can see that x^m is a common factor, dropping it gives:

$$m(m-1)+m+1=0 \Leftrightarrow m^2-\cancel{m}+\cancel{m}+1=0 \Leftrightarrow m^2+1=0$$

So $y=x^m$ is a solution of the given ODE if m is a root of the equation

Let's find the root of the equation

$$m^2+1=0 \Leftrightarrow m^2-i^2=0 \Leftrightarrow (m-i)(m+i)=0$$

So, it has the complex conjugate roots:

$$m_1=i \quad \wedge \quad m_2=-i$$

Now, we use that fact $x=e^{\ln x}$:

$$x^{m_1}=x^i=(e^{\ln x})^i=e^{i\ln x}$$

$$x^{m_2}=x^{-i}=(e^{\ln x})^{-i}=e^{-i\ln x}$$

Recall that

$$e^x=e^{a+ib}=e^a(\cos b+i\sin b)$$

So, we have:

$$e^{i\ln x}=e^0(\cos(\ln x)+i\sin(\ln x))=\cos(\ln x)+i\sin(\ln x)$$

and

$e^{-i\ln x} = e^0(\cos(\ln x) - i\sin(\ln x)) = \cos(\ln x) - i\sin(\ln x)$

that gives

$$x^{m_1} = \cos(\ln x) + i\sin(\ln x)$$
$$x^{m_2} = \cos(\ln x) - i\sin(\ln x)$$

Adding this two formula gives

$$x^{m_1} + x^{m_2} = \cos(\ln x) + \cancel{i\sin(\ln x)} + \cos(\ln x) - \cancel{i\sin(\ln x)}$$
$$= 2\cos(\ln x)$$

Now, divide it by 2:

$$\frac{x^{m_1} + x^{m_2}}{2} = \frac{\cancel{2}\cos(\ln x)}{\cancel{2}} = \cos(\ln x)$$

Next, subtract the second formula from the first ~~and~~ and divide it by to $2i$ after that

$$x^{m_1} - x^{m_2} = \cancel{\cos(\ln x)} + i\sin(\ln x) - \cancel{\cos(\ln x)} + i\sin(\ln x))$$
$$= 2i\sin(\ln x)$$

Divide it by 2

$$\frac{x^{m_1} - x^{m_2}}{2i} = \frac{\cancel{2i}\sin(\ln x)}{\cancel{2i}} = \sin(\ln x)$$

By the superposition principle, $\cos(\ln x)$ and $\sin(\ln x)$ are the solution of the Euler-Cauchy equation. Their quotient is not constant, so the solutions

$$y_1 = \cos(\ln x) \quad \& \quad y_2 = \sin(\ln x)$$

are linearly independent and form a basis of solution. So the general ~~fo~~ solution is:

$$y = c_1 y_1 + c_2 y_2 = \underline{c_1\cos(\ln x) + c_2\sin(\ln x)}$$

$$\Rightarrow y' = -c_1\sin(\ln x)\,(\ln x)' + c_2(\ln x)\cdot(\ln x)'$$

$$= -\frac{c_1}{x}\sin(\ln x) + \frac{c_2}{x}\cos(\ln x)$$

Now determining c_1 and c_2 from IVP

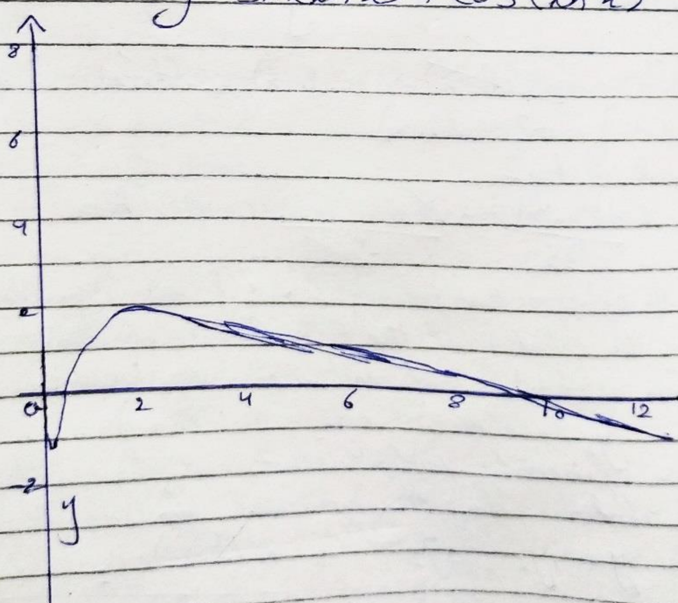
$$\begin{cases} 1 = y(1) = c_1 \cos(\ln 1) + c_2 \sin(\ln 1) \\ 1 = y'(1) = -c_1 \sin(\ln 1) + 3c_2 \cos(\ln 1) \end{cases}$$

$$\Rightarrow \begin{cases} 1 = c_1 \cos(0) + c_2 \sin(0) \\ 1 = -c_1 \sin(0) + c_2 \cos(0) \end{cases}$$

$$\Rightarrow \begin{cases} 1 = c_1 \\ 1 = c_2 \end{cases}$$

The particular solution of IVP is

$$y = \sin(\ln x) + \cos(\ln x)$$



(15)

18 $(9x^2D^2+3xD+I)y=0$, $y(1)=1$, $y'(1)=0$

Solution:

First we need to apply the given operator to the function:

$$9x^2D^2y+3xDy+Iy=9x^2D(Dy)+3xDy+y$$
$$=9x^2y''+3xy'+y$$

Lets Solve:

$$9x^2y''+3xy'+y=0$$

Lets substitute:

$y=x^m$, $y'=mx^{m-1}$, $y''=m(m-1)x^{m-2}$ into the given ODE, which gives:

$$9x^2m(m-1)x^{m-2}+3xmx^{m-1}+x^m=0$$
$$9\cancel{x^2}m(m-1)x^m\cdot\cancel{x^{-2}}+3\cancel{x}mx^m\cdot\cancel{x^{-1}}+x^m=0$$

We can see that x^m is a common factor, dropping it gives

$9m(m-1)+3m+1=0 \Leftrightarrow 9m^2-9m+3m+1=0 \Leftrightarrow 9m^2-6m+1=0$

So, $y=x^m$ is a solution of the given ODE if m is a root of the equation

Lets find the roots of the equation:

$$m^2-4m+4=0 \Leftrightarrow (m-2)^2=0$$

$$9m^2-6m+1=0 \Leftrightarrow m_{\frac{1}{2}}=\frac{6\pm\sqrt{6^2-4\cdot 9}}{18}$$

$$m_{\frac{1}{2}}=\frac{6}{18}=\frac{1}{3}$$

So, it has real double root:

$$m=\frac{1}{3}$$

Real double root m provides a real solution

$$y_1=x^m=x^{\frac{1}{3}}$$

We can find a second linearly independent solution y_2 using the method of reduction of order

First, we need to write the given ODE in standard form

$$y'' + \frac{1}{3x} \cdot y' + \frac{1}{9x^2} \cdot y = 0$$

Now, we can see that:

$$p(x) = \frac{1}{3} \cdot \frac{1}{x} \Rightarrow \int p\,dx = \frac{1}{3} \ln|x|$$

Let's find U:

$$e^{-\int p dx} = e^{-\frac{1}{3} \cdot \ln|x|} = \left(e^{\ln|x|}\right)^{-\frac{1}{3}} = x^{-\frac{1}{3}}$$

$$\Rightarrow U = x^{-\frac{1}{3}} \cdot \frac{1}{\left(x^{\frac{1}{3}}\right)^2} = x^{-\frac{1}{3} - \frac{2}{3}} = x^{-1} = \frac{1}{x}$$

By integration, we have

$$u = \int \frac{dx}{x} = \ln|x|$$

So

$$y_2 = u y_1 = y_1 \ln x = x^{\frac{1}{3}} \ln x$$

Since their quotient is not constant, y_1 and y_2 are linearly independent and constitute a basis of solutions for the given ODE, for all x for which $y_1, y_2 \in \mathbb{R}$

So, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 x^{\frac{1}{3}} + x^{\frac{1}{3}} \ln x$$

$$= x^{\frac{1}{3}} (c_1 + c_2 \ln x)$$

Product rule.

$$\Rightarrow y' = \left(x^{\frac{1}{3}}\right)' (c_1 + c_2 \ln x) + x^{\frac{1}{3}} (c_1 + c_2 \ln x)'$$

$$= \frac{1}{3} \cdot x^{-\frac{2}{3}} (c_1 + c_2 \ln x) + x^{\frac{1}{3}} c_2 \cancel{x} \cdot \frac{1}{x}$$

$$= \frac{1}{3} \cdot x^{-\frac{2}{3}} (c_1 + c_2 \ln x) + x^{\frac{1}{3}} c_2$$

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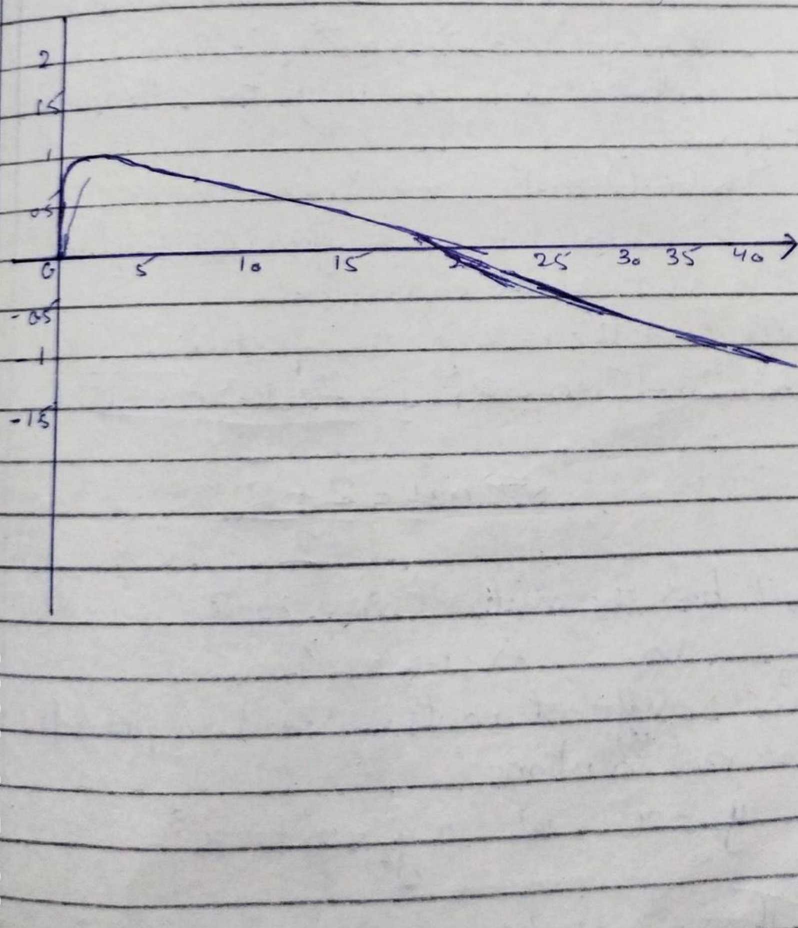
Now we determine c_1 and c_2 from IVP.

$$\begin{cases} 1 = y(1) = 1^{\frac{1}{3}}(c_1 + c_2 \ln 1) \\ 0 = y'(1) = \frac{1}{3} \cdot 1^{-\frac{2}{3}}(c_1 + c_2 \ln 1) + 1^{\frac{1}{3}} c_2 \end{cases} \Rightarrow \begin{cases} 1 = c_1 \\ 0 = \frac{c_1}{3} + c_2 \end{cases}$$

$$\Rightarrow \begin{cases} 1 = c_1 \\ -\frac{1}{3} = c_2 \end{cases}$$

The particular solution of IVP is

$$y = x^{\frac{1}{3}}\left(1 - \frac{1}{3}\ln x\right)$$



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19 $(x^2D^2 - xD - 15I)y = 0,\ y(1) = 0.1,\ y'(1) = -4.5$

Solution:

First, we need to apply the given operator to the given function

$$x^2D^2y - xDy - 15Iy = x^2D(Dy) - xDy - 15y$$
$$= x^2y'' - xy' - 15y$$

Let's solve the equation,

$$x^2y'' - xy' - 15$$

Let's substitute:

$$y = x^m,\ y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$$

into the given ODE. This gives:

$$x^2m(m-1)x^{m-2} - xmx^{m-1} - 15x^m = 0$$
$$\cancel{x^2}m(m-1)x^m\cancel{x^{-2}} - \cancel{x}mx^m\cancel{x^{-1}} - 15x^m = 0$$

We see that x^m is a common factor, dropping it gives

$$m(m-1) - m - 15 = 0 \Leftrightarrow m^2 - 2m - 15 = 0$$

So, $y = x^m$ is a solution of the given ODE if m is a root of the equation

Let's find the roots of the equation

$$m^2 - 2m - 15 = 0 \Leftrightarrow m_{\frac{1}{2}} = \frac{2 \pm \sqrt{(-2)^2 + 4 \cdot 15}}{2}$$

$$\Leftrightarrow m_{\frac{1}{2}} = \frac{2 \pm 8}{2}$$

So, it has the distinct real roots:

$$m_1 = 5 \ \wedge\ m_2 = -3$$

Real different roots m_1 and m_2 provide two real solutions

$$y_1 = x^{m_1} = x^5 \ \wedge\ y_2 = x^{m_2} = x^{-3}$$

Their quotient is not constant, so the solutions y_1 and y_2 are linearly independent and constitute a basis of solutions for the given ODE, for all x for which

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$y_1, y_2 \in R$

So, the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 x^5 + c_2 x^{-3}$$

$$\Rightarrow y' = 5c_1 x^4 - 3c_2 x^{-4}$$

Now, all we need to do is to determine c_1 and c_2 from IVP

$$\begin{cases} 0.1 = y(1) = c_1 \cdot 1^5 + c_2 \cdot 1^{-3} \\ -4.5 = y'(1) = 5c_1 \cdot 1^4 - 3c_2 \cdot 1^{-4} \end{cases} \Rightarrow \begin{cases} 0.1 = c_1 + c_2 \\ -4.5 = 5c_1 - 3c_2 \end{cases}$$

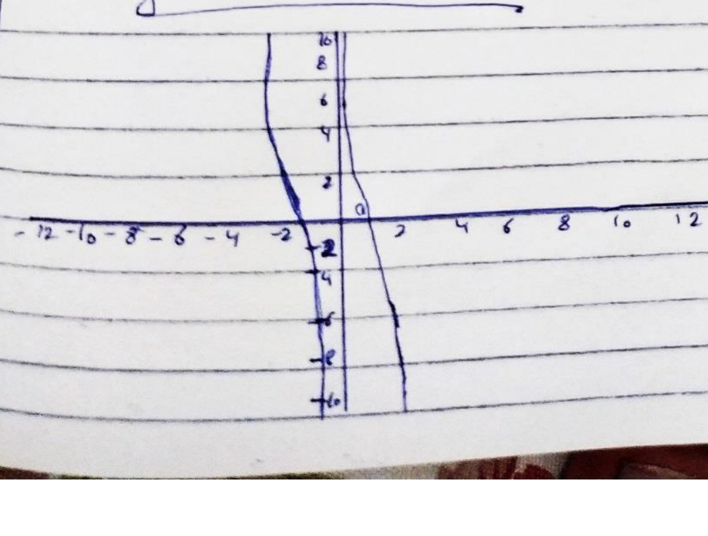
$$\Rightarrow \begin{cases} 0.1 - c_2 = c_1 \\ -4.5 = 5(0.1 - c_2) - 3c_2 \end{cases} \Rightarrow \begin{cases} 0.1 - c_2 = c_1 \\ 5 = 8c_2 \quad /:8 \end{cases}$$

$$\Rightarrow \begin{cases} 0.1 - c_2 = c_1 \\ 0.625 = c_2 \end{cases} \Rightarrow \begin{cases} 0.1 - 0.625 = c_1 \\ 0.625 = c_2 \end{cases}$$

$$\Rightarrow \begin{cases} -0.525 = c_1 \\ 0.625 = c_2 \end{cases}$$

The particular solution of the IVP is:

$$y = -0.525x^5 + 0.625x^{-3}$$



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Use the method of separation of variables to find the general solution to the following differential equations.

$x' = \sqrt{x}$

Solution:

$$x' = \sqrt{x}$$

$$\frac{dx}{dt} = \sqrt{x}$$

seperating variables.

$$\frac{dx}{\sqrt{x}} = dt$$

Integrating

$$\int \frac{dx}{\sqrt{x}} = \int dt$$

$$\int x^{-\frac{1}{2}} dx = \int dt$$

$$\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = t + C \quad \text{where } C \text{ is constant}$$

$$\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = t + c$$

$$2x^{\frac{1}{2}} = t + C$$

$$2x^{\frac{1}{2}} = t + c$$

$$x^{\frac{1}{2}} = \frac{t+c}{2}$$

$$x = \left(\frac{t+C}{2}\right)^2$$ is the general solution.

b $x' = e^{-2x}$

Solution:

$$\frac{dx}{dt} = e^{-2x}$$

seperating the variables

$$\frac{dx}{e^{-2x}} = dt$$

$$e^{2x}dx = dt$$

Integrating

$$\int e^{2x}dx = \int dt$$

$$\frac{e^{2x}}{2} = t+c$$

$$e^{2x} = 2(t+c)$$

$$\cancel{\Rightarrow e^{2x} = 2}$$

$$\Rightarrow \ln(e^{2x}) = \ln[2\{t+c\}]$$

$$\Rightarrow 2x \ln e = \ln[2\{t+c\}]$$

$$\Rightarrow 2x = \ln[2\{t+c\}]$$

$$\Rightarrow x = \frac{1}{2}\ln[2\{t+c\}]$$ is the G. Solution.

c $y' = 1+y^2$

Solution:

$$\frac{dy}{dt} = 1+y^2$$

Seperating the variables.

$$\frac{dy}{1+y^2} = dt$$

Integrating

$$\int \frac{dy}{1+y^2} = \int dt$$

$\Rightarrow \tan^{-1} y = t + c$

$\Rightarrow y = \tan(t+c)$ is the general solution.

d. $u' = \frac{1}{5-2u}$

Solution:

$$\frac{du}{dt} = \frac{1}{5-2u}$$

Seperating variables

$$(5-2u)\,du = dt$$

$$5du - 2u\,du = dt$$

Integrating

$$5\int du - 2\int u\,du = \int dt.$$

$$5u - \frac{2u^{1+1}}{1+1} = t+c$$

$$5u - \frac{2u^2}{2} = t+c$$

$$5u - u^2 = t+c$$

is the general solution

e. $x' = ax+b$, $a, b > 0$

Solution:-

$$\frac{dx}{dt} = ax+b$$

Seperating variables.

$$\frac{dx}{ax+b} = dt$$

Integrating

$$\int \frac{dx}{ax+b} = \int dt$$

$$\Rightarrow \frac{1}{a}\int \frac{a\,dx}{ax+b} = \int dt$$

$$\frac{1}{a}\ln|ax+b| = t+c$$

$$\ln|ax+b| = a(t+c)$$

$$ax+b = e^{a(t+c)}$$

$$ax = e^{a(t+c)} - b$$

$$x = \frac{e^{a(t+c)} - b}{a}$$ is the general Solution.

f $Q' = \frac{Q}{4+Q^2}$

Solution:

$$\frac{dQ}{dt} = \frac{Q}{4+Q^2}$$

separating variables.

$$\Rightarrow \frac{4+Q^2}{Q}\,dQ = dt$$

$$\Rightarrow \frac{4}{Q}\,dQ + \frac{Q^2}{Q}\,dQ = dt$$

$$\Rightarrow \frac{4}{Q}\,dQ + Q\,dQ = dt$$

Integrating

$$\Rightarrow \int \frac{4}{Q}\,dQ + \int Q\,dQ = \int dt$$

$$\Rightarrow 4\ln Q + \frac{Q^2}{2} = t + C$$ is the general Solution.

g $x' = e^{x^2}$

Solution:

(h) $y' =$

$x' = e^{x^2}$

Solution:

$$\frac{dx}{dt} = e^{x^2}$$

separating variables.

$$\frac{dx}{e^{x^2}} = dt$$

$$e^{-x^2} dx = dt$$

Integrating

$$\int e^{-x^2} dx = dt \quad -①$$

let

$$-x^2 = p \rightarrow ②.$$

$$2x dx = dp$$

$$dx = \frac{-dp}{2x}$$

$$dx = \frac{-dp}{2\sqrt{p}} \quad -③.$$

$$= -\int \frac{e^{p} dp}{2\sqrt{p}} = dt$$

$$= \frac{-1}{2} \int e^{+p} p^{-\frac{1}{2}} dp = dt$$

$$= \int e^{+p} p^{-\frac{1}{2}} dp = 2(t+c)$$

$$= \frac{1}{2} x e^{-x^2} = 2(t+c)$$ is the general solution.

$y' = r(a-y)$

Solution:

$$\frac{dy}{dt} = r(a-y)$$

separating variables.

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$dy = r(a-y)dt$

$\frac{dy}{a-y} = r\,dt$

Integrating

$\int \frac{dy}{a-y} = r\int dt$

$\Rightarrow -\int \frac{-dy}{a-y} = r\int dt$

$\Rightarrow -\ln|a-y| = rt + c$

$\Rightarrow \ln|a-y|^{-1} = rt + c$

$\Rightarrow \ln\left|\frac{1}{a-y}\right| = rt + c$

$\Rightarrow \frac{1}{a-y} = e^{rt+c}$

$1 = (a-y)(e^{rt}+c)$

$(a-y) = \frac{1}{(e^{rt}+c)}$

$y = a - \frac{1}{e^{rt}+c}$ is the general solution

Q2 Solve $y' =$ constants:

Solution:

sep

In

2 Solve $y' = r(a-y)$ where r and a are constants.

Solution:

$$\frac{dy}{dt} = r(a-y)$$

separating variables.

$$dy = r(a-y)\,dt$$

$$\frac{dy}{a-y} = r\,dt$$

Integrating

$$\int \frac{dy}{a-y} = r\int dt$$

$$\Rightarrow -\int \frac{-dy}{a-y} = r\int dt$$

$$\Rightarrow -\ln|a-y| = rt + c$$

$$\Rightarrow \ln|a-y|^{-1} = rt + c$$

$$\Rightarrow \ln\left|\frac{1}{a-y}\right| = rt + c$$

$$\Rightarrow \frac{1}{a-y} = e^{rt} + c$$

$$1 = (a-y)(e^{rt} + c)$$

$$(a-y) = \frac{1}{(e^{rt} + c)}$$

$$y = a - \frac{1}{e^{rt} + C}$$ is the general solution.

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3 In Exercises 1(a) & (b) find the solution to the resulting IVP when $x(0)=1$.

Solution:

General Solution of 1(a)

$$x = \left(\frac{t+c}{2}\right) \rightarrow (1)$$

when $x(0)=1$

$t=0, x=1$

put in eq (1)

$$1 = \left(\frac{0+c}{2}\right)^2$$

$$\Rightarrow 1 = \frac{c}{4}$$

$$\Rightarrow c = 4$$

then one solution become

$$x = \left(\frac{t+4}{2}\right)^2$$

part b:

Gr. solution:

$$x = \frac{1}{2} \ln[2\{t+c\}] - (1)$$

where $x(0)=1 \Rightarrow t=0, x=1$

then equation (1) becomes

$$1 = \frac{1}{2} \ln[2\{t+c\}]$$

$$2 = \ln[2c]$$

$$e^2 = 2c \Rightarrow c = \frac{1}{2}e^2$$

putting the values of C in eq (1) we have

4. Find th

(a) $x' = \frac{2x}{t}$

solution

(b) $\theta' =$

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~~putting~~ $x = \frac{1}{2} \ln\left[2\left\{t + \frac{1}{2} e^{2}\right\}\right]$

Find the general solution.

(a) $x' = \frac{2x}{t+1}$

solution:

$$\frac{dx}{dt} = \frac{2x}{t+1}$$

separating variables:

$$\frac{dx}{2x} = \frac{dt}{t+1}$$

Integrating

$$\frac{1}{2}\int \frac{dx}{x} = \int \frac{dt}{t+1}$$

$$\frac{1}{2} \ln x = \ln|t+1| + \ln C$$

$$\ln \sqrt{x} = \ln|t+1| + \ln C$$

$$\ln\sqrt{x} = \ln(|t+1|)(c)$$

$$\sqrt{x} = (t+1)(c)$$

$x = (c(t+1))^2$ is the general solution

(b) $\dot{\theta} = t\sqrt{t^2+1} \sec\theta$

Solution:—

$$\frac{d\theta}{dt} = t\sqrt{t^2+1} \sec\theta$$

$$\frac{d\theta}{\sec\theta} = t\sqrt{t^2+1}\, dt$$

Integrating

$$\int \frac{d\theta}{\sec\theta} = \int t\sqrt{t^2+1}\, dt.$$

$$\Rightarrow \int \cos\theta \, d\theta = \frac{1}{2}\int \sqrt{t^2+1}\; 2t\, dt$$

$$\Rightarrow \sin\theta = \frac{1}{2}\frac{(t^2+1)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$\Rightarrow \sin\theta = \frac{1}{3}(t^2+1)^{\frac{3}{2}} + C$$ is the general solution.

ⓒ $(2u+1)u' - (t+1) = 0$

solution:

$$(2u+1)\frac{du}{dt} = t+1$$

$$(2u+1)\,du = (t+1)\,dt$$

Integrating

$$u^2 + u = \frac{t^2}{2} + t + C$$ is the general solution.

ⓓ $R' = (t+1)(R^2+1)$

solution:

$$\frac{dR}{dt} = (t+1)(R^2+1)$$

$$\frac{dR}{R^2+1} = (t+1)\,dt$$

Integrating

$$\tan^{-1}R = \frac{t^2}{2} + t + C$$

$$R = \tan\left(\frac{t^2}{2} + t + C\right)$$ is the general solution.

ⓔ $y' + y + \frac{1}{y}$

solution

ⓕ $(t+1)$

solut

e) $y'y + \frac{1}{y} = 0$

Solution

$$y' = -\left(y + \frac{1}{y}\right)$$

$$\frac{dy}{dt} = -\left(\frac{y^2+1}{y}\right)$$

$$\frac{y}{y^2+1}\,dy = -dt$$

Integrating

$$\frac{1}{2}\int \frac{2y\,dy}{y^2+1} = -\int dt$$

$\Rightarrow \frac{1}{2}\ln|y^2+1| = -t + c$ is the general solution.

f) $(t+1)x' + x^2 = 0$

Solution:

$$(t+1)\frac{dx}{dt} = x^2$$

$$\frac{dx}{-x^2} = \frac{1}{t+1}\,dt$$

$$-\int \frac{dx}{x^2} = \int \frac{dt}{t+1}$$

$$-\int x^{-2}\,dx = \int \frac{dt}{t+1}$$

$$\frac{-x^{-1}}{1} = \ln(t+1) + \ln C$$

$$\frac{1}{x} = \ln C(t+1)$$

$x = \frac{1}{\ln C(t+1)}$ is the general solution.