Q1: Part 'A'
Define DE with two examples?

A differential equation is an equation involving and it's derivatives.
The solution to a differential equation is in the form of a function/class of a function.

Examples of D.E:

$\Rightarrow \dfrac{dy}{dx} + y^2 x = 2x$

$\Rightarrow \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0.$

Q1: Part 'B'

Define a Seperable Differential Equation?

A seperable differential equation is any equation that we can write in the following form.

$$N(y)\frac{dy}{dx} = M(x)$$

i, Solve the following Initial Value Problem using Seperable DE and find the interval of validity of the solution.

a/ $y' = \frac{xy^3}{\sqrt{1+x^2}}$; $y(0) = -1$.

$$y^{-3}dy = x(1+x^2)^{-1/2}dx$$

now integrating both sides.

$$\int y^{-3}dy = \int x(1+x^2)^{-1/2}dx$$

$$-1/2y^2 = \sqrt{1+x^2} + c.$$

Apply the initial condition to get the value of c.

$$-\frac{1}{2} = \sqrt{1} + c$$

$$c = -3/2$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now solving for $y(x)$.

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

$$y(x) = \pm \frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Reapplyin the initial condition shows us that "$-$" is the correct sign, The explicit solution is then

$$y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Lets get the interval of validity now,

$$1+x^2 \geq 0 \text{ (the internal root)}$$

$$3-2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4(1+x^2)$$

$$\frac{9}{4} > 1+x^2$$

$$\frac{5}{4} > x^2$$

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

This also contains the initial condition $x=0$. This interval is the interval of validity.

Ans

Q1 => b => i => b.

i, => "b" part

$$y' = e^{-y}(2x-4) \qquad y(5) = 0.$$

Seperating and integrating both sides:

$$e^{y}\,dy = (2x-4)\,dx$$

$$\int e^{y}\,dy = \int (2x-4)\,dx$$

$$e^{y} = x^2 - 4x + c$$

Applying the initial condition:

$$1 = 25 - 20 + c$$

$$c = -4$$

The implicit solution is then:

$$e^{y} = x^2 - 4x - 4$$

Taking natural log on both sides to find explicit solution:

$$y(x) = \ln(x^2 - 4x - 4).$$

Now finding interval of validity

$$x^2-4x-4>0$$

the quadratic will be zero at the two point $x=2\pm2\sqrt{2}$.

So, the possible intervals of validity are:

$$\infty < x < 2-2\sqrt{2}$$
$$2+2\sqrt{2} < x < \infty$$

as we know from the initial condition that $x=5$

The interval of validity is

$$2+2\sqrt{2} < x < \infty$$

Ans

Q2, Part 'A'

i, Explain the steps for solving Linear Differential Equation.

Steps for solving Linear Differential is as follows:

=> Put the differential equation in the correct initial form.
=> Find the integrating factor, $\mu(t)$ ∴ $\mu(t) = e^{\int p(t)dt}$.
=> Multiply everything in the differential equation by $\mu(t)$ and verify that the left side becomes the product rule $(\mu(t)y(t))'$ and write it as such.
=> Integrate both sides, make sure you properly deal with the constant of integration.

⇒ Solve for the solution $y(t)$

ii) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1.$

$$y\left[\frac{\pi}{4}\right] = 3\sqrt{2}, \quad 0 \le x \le \frac{\pi}{2}.$$

Rewriting the differential equation to get the coefficient of the derivative one.

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}.$$

$$y' + \tan(x)y = 2\cos^2(x)\sin(x) - \sec(x) \quad \text{eq (i)}.$$

Now find the integrating factor:

$$\mu(t) = e^{\int \tan(x)dx} = e^{\ln|\sec(x)|} = e^{\ln \sec(x)} = \sec(x)$$

$$\int \tan(x)\,dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1} = \ln|\sec(x)|$$

Multiplying integrating factor with eqⁿ (ii).

$$\sec(x)\,y' + \sec(x)\tan(x)\,y = 2\sec(x)\cos^2(x)\sin x - \sec^2(x).$$

$$(\sec(x)\,y)' = 2\cos(x)\sin x - \sec^2(x)$$

Integrate both sides:

$$\int(\sec(x)\,y(x))'\,dx = \int 2\cos(x)\sin(x) - \sec^2(x)\,dx$$

$$\sec(x)\,y(x) = \int \sin(2x) - \sec^2(x)\,dx.$$

$$\sec(x)\,y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + c$$

Now using trig formula's:

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x).$$

Now, apply the initial condition to find the value of c.

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin(\pi/4) + c\cos(\pi/4)$$

$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2}$$

$$c = 7.$$

The solution is then,

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x).$$

iii) $x' + 2x = \sin t$.

$$\frac{dy}{dx} + 2x = \sin t \quad \text{— eq (i)}$$

Integrating factor:

$$= e^{\int 2(x)dx}$$

$$= e^{x^2}$$

Multiplying the I.F with eq (i)

$$e^{x^2}\frac{dy}{dx} + e^{x^2}2x = e^{x^2}\sin t.$$

$$x^2\frac{dy}{dx} + x^2 2x = x^2 \sin t.$$

$$\frac{d}{dx}(e^{x^2}y) = \int e^{x^2}\sin t$$

$$x^2 y = \ln[x^3 \sin(t) - \frac{2}{3}x^3 \sin(t) + c]$$

$$y = \frac{1}{x^2} \ln [x^3 \sin(t) - \frac{2}{3} x^3 \sin(t) + c],$$

Q3: Solve the following IVP for the exact equation and find the interval of validity for the solution.

i, $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0 \quad y(0) = -3$

First we will identify M & N and check that the equation is exact.

$$M = 2xy - 9x^2 \qquad M_y = 2x$$
$$N = 2y + x^2 + 1 \qquad N_x = 2x$$

it is exact equation

Now,

$$\psi_x = M$$
$$\psi_y = N$$

$$\psi = \int M\,dx \quad \text{or} \quad \psi = \int N\,dy.$$

Now,

$$\psi(x,y) = \int 2xy - 9x^2\,dx$$

$$= x^2y - 3x^2 + h(y).$$

We used $\psi_x = M$ to find most of $\psi(x,y)$ so we'll use $\psi_y = N$ to find $h(y)$.

$$\psi_y = x^2 + h'(y)$$

$$= 2y + x^2 + 1 \quad = N$$

From this we can see that

$$h'(y) = 2y + 1.$$

Now we can find $h(y)$ by integrating

$$h(y) = \int 2y + 1\,dy$$

$$= y^2 + y + k$$

So, we can write down $\Psi(x,y)$.

$$\Psi(x,y) = x^2y - 3x^3 + y^2 + y + k$$

$$= y^2 + (x^2+1)y - 3x^2 + k$$

Now,

$$y^2 + (x^2+1)y - 3x^2 + k = c$$

$$y^2 + (x^2+1)y - 3x^2 = c - k$$

$$y^2 + (x^2+1)y - 3x^2 = c$$

Now we will apply initial condition to find c.

$$(-3)^2 + (0+1)(-3) - 3(0)^3 = c$$

$$c = 6.$$

The implicit solution is then

$$y^2 + (x^2+1)y - 3x^3 - 6 = 0.$$

By applying quadratic formula:

$$y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^3-6)}}{2(1)}$$

Reapplying the initial conditions to find out which of the two signs ± we need

$$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = -3, 2$$

so "−" is sign we need to use

$$y(x) = \frac{-(x^2+1) - \sqrt{x^4+12x^3+2x^2+25}}{2}$$

For the interval of validity we will solve

$$x^4+12x^3+2x^2+25 = 0.$$

upon solving this equation is zero at $x = -11.8155$ and $x = -1.3969$.

The possible two intervals are

$$-\infty < x \leq -11.8155$$
$$-1.3969 \leq x < \infty$$

However, the intervals of validity need to be continuous intervals and contain the value of x that is used in initial condition.

Therefore the interval of validity must be:

$$-1.3969 \leq x < \infty.$$

ii, $\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0$

$y(5) = 0.$

First we will deal with the sign "-" seperating the two terms.

$\frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)y' = 0.$

Now,

$\left\{ \cancel{M = \frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)y'} \right\}$

$M = \frac{2ty}{t^2+1} - 2t \qquad M_y = \frac{2t}{t^2+1}$

$N = \ln(t^2+1) - 2 \qquad N_t = \frac{2t}{t^2+1}$

This is exact equation and we will integrate the first one.

$$\psi(t,y) = \int \frac{2ty}{t^2+1} - 2t\,dt = y\ln(t^2+1) - t^2 + h(y)$$

Differentiate with respect to y and compare to N.

$$\psi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2 = N.$$

$$h'(y) = -2 \Rightarrow h(y) = -2y.$$

This gives us

$$\psi(t,y) = y\ln(t^2+1) - t^2 - 2y$$

The implicit solution is then

$$y\ln(t^2+1) - t^2 - 2y = c.$$

Applying the initial condition gives

$$c = -25.$$

The implicit solution is now,

$y(\ln(t^2+1)-2)-t^2=-25$.

now we need to solve for y.

$$y(t)=\frac{t^2-25}{\ln(t^2+1)-2}$$

Now to find interval of validity:

$$\ln(t^2+1)-2=0$$
$$\ln(t^2+1)=2$$
$$t^2+1=e^2$$
$$t=\pm\sqrt{e^2-1}$$

We now have three possible intervals:

$$-\infty<t<-\sqrt{e^2-1}$$
$$-\sqrt{e^2-1}<t<\sqrt{e^2-1}$$
$$\sqrt{e^2-1}<t<\infty$$