Differential Equation.

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Date: 13 April, 2020.

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Q1) Define differentital equation along with 2 examples?

A) Differentital Equation:-

A differentital equation is any equation which contain derivatives, either ordinary derivatives or Partial derivative

"Or"

An ordinary Differentital equation or just differentital is another type of equation, where unknown is not a number but a function.

Example 1:-

Show that $y(x) = x^{-3/2}$ is solution to $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$.

Sol:-

We will need first and second derivative, to do this.

$$y'(x) = \frac{-3}{2} x^{-5/2} \quad , \quad y''(x) = \frac{15}{4} x^{-7/2}$$

Now, Putting value into equation.

$$= 4x^2 \left(\frac{15}{4} x^{-7/2} \right) + 12x \left(-\frac{3}{2} x^{-5/2} \right) + 3\left(x^{-3/2} \right)$$

$$= 15x^{-3/2} - 18x^{-3/2} + 3x^{-3/2} = 0$$

$$0 = 0$$

So,

$y(x) = x^{-3/2}$ satisfy the differentital equation, and hence it's solution.

$$y(x) = x^{-3/2} = \frac{1}{\sqrt{x^3}}.$$

Example 2:

$y(x) = x^{-3/2}$ is solution to $4x^2y'' + 12xy' + 3y = 0$, $y(4) = \frac{1}{8}$, and $y'(4) = \frac{-3}{64}$.

Sol:.

$$y(4) = 4^{-3/2} = \frac{1}{(\sqrt{4})^3} = \frac{1}{8}$$

$$y'(4) = \frac{-3}{2} 4^{-5/2} = \frac{-3}{2} \frac{1}{(\sqrt{4})^5} = \frac{-3}{64}.$$

and so, solution meets it's initial condition $y(4) = \frac{1}{8}$ and $y'(4) = \frac{-3}{64}$. In fact, $y(x) = x^{-3/2}$,

which satitisfies both condition.

(b) Define a Separable Differentital Equation?

(A) Separable Differentibal Equation:

A separable differentital equation is that we can write in following form.

$$N(y)\frac{dy}{dx} = M(x).$$

To solve it, we need to integrate b.s w.r.t "x"

$$\int N(y)\frac{dy}{dx}\,dx = \int M(x)\cdot dx.$$

Now remember, y is really y(x), and we can use following subsuibtion.

$$u = y(x),\quad du = y'(x)\,dx = \frac{dy}{dx}\,dx.$$

Applying this subsuibtion to integeral, we get

$$\int N(u)\,du = \int M(x)\,dx.$$

finally

$$N(y)\,dy = M(x)\,dx.$$

1. Solve following Initial value problem (IVP) using separable D.E and find interval of validity of solution.

(a) $y' = \frac{xy^3}{\sqrt{1+x^2}}$, $y(0) = -1$

Sol:

first separate and integrate b.s

$$y^{-3}\,dy = x(1+x^2)^{-1/2}\,dx$$

$$\int y^{-3}\,dy = \int x(1+x^2)^{-1/2}\,dx$$

$$\frac{-1}{2y^2} = \sqrt{1+x^2} + c$$

Apply the condition to get c.

$$\frac{-1}{2} = \sqrt{1} + c \quad , \quad c = \frac{-3}{2}$$

then,

$$\frac{1}{-2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now, solve for $y(x)$.

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

$$y(x) = \pm \frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Replacing the sign $-$, to find explicit solution.

$$y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

first, since $1+x^2 \geq 0$,

$$3-2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4(1+x^2)$$

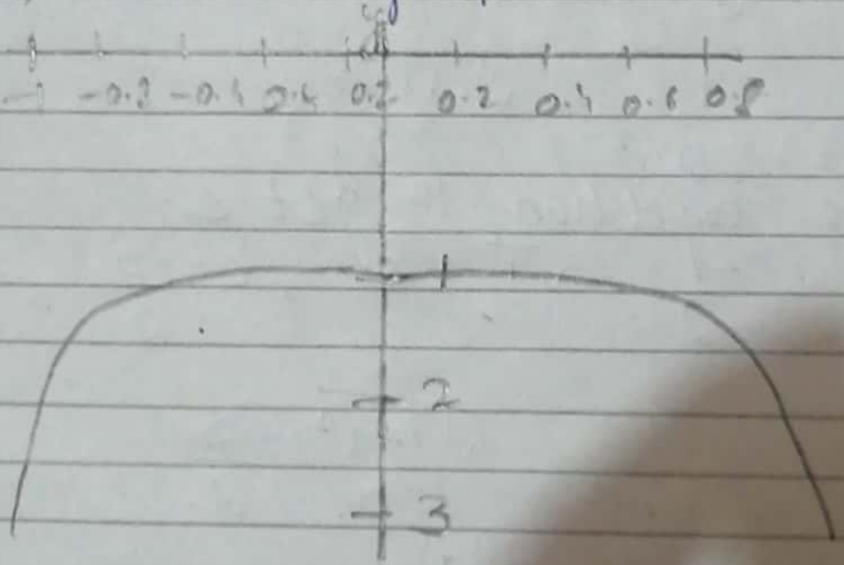
$$\frac{9}{4} > 1+x^2$$

$$\frac{5}{4} > x^2$$

So, we can square b.s because of inequality to be positived in this case,

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

$x = 0$, Therefore graph of solution



(5) $y' = e^{-y}(2x-4)$, $y(5) = 0$.

Sol.

It's easy to separate integrate b.s.

$$e^{y}\,dy = (2x-4)\,dx$$

$$\int e^{y}\,dy = \int (2x-4)\,dx$$

$$e^{y} = x^2 - 4x + c$$

(7) (9)

Apply initial condition gives

$$1 = 25 - 20 + c \quad , c = -4.$$

then implicit solution

$$e^y = x^2 - 4x - 4.$$

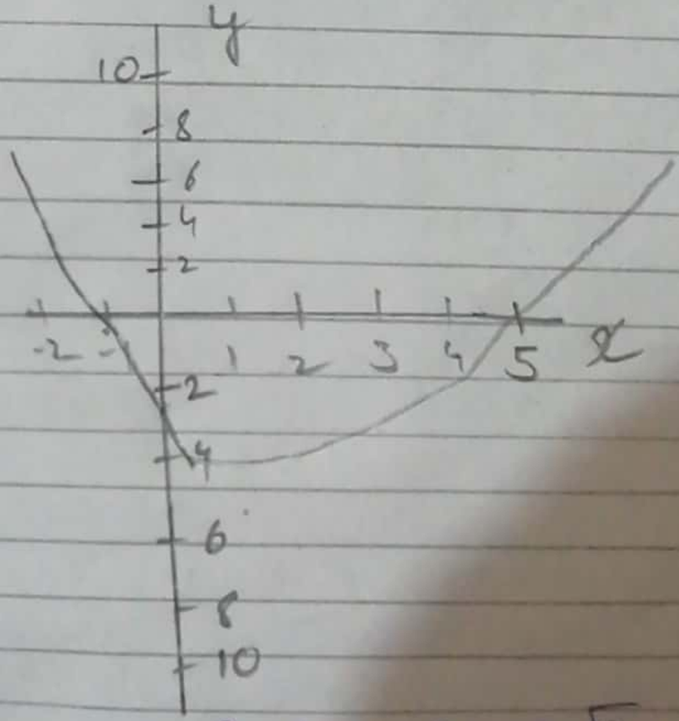
Taking log on b.s

$$y(x) = \ln(x^2 - 4x - 4).$$

Now interval of validility, last step,

$$x^2 - 4x - 4 > 0$$

The quadratic will be 0 at two points $x = 2 \pm 2\sqrt{2}$.



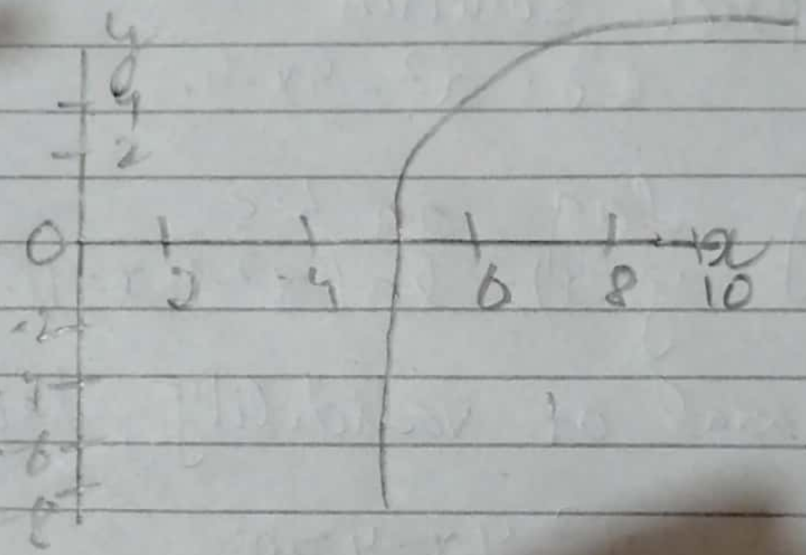
$$-\infty < x < 2 - 2\sqrt{2}$$

$$2 + 2\sqrt{2} < x < \infty$$

from graph we see, $x = 5$.

$$2 + 2\sqrt{2} < x < \infty.$$

Here is graph of solution.



2) Solve the follaving IVP using linere Differentibal method.

1) Explain steps for solving Linear Differentital Equation.

* First order:

They are first order, where there is only

$$\frac{dy}{dx}, \text{ not } \frac{d^2y}{dx^2} \text{ or } \frac{d^3y}{dx^3}$$

* Linear:

A first order differential equation is linear, and look like

$$\frac{dy}{dx} + P(x)y = Q(x).$$

where $P(x)$ & $Q(x)$ are function x, and we two new function called them u, v and equal to $y = uv$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Steps:

1. subsitute $y = uv$ and

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

into

$$\frac{dy}{dx} + P(x)y = Q(x).$$

2. Factor parts involing v.
3. Put them to equal 0.
4. Solve using separation of variable to find u.
5. subsuite u to into back equation.
6. solve to find v.
7. finally, subsuite u and v into y, uv to

get solution.

(ii) $\cos(x)y' + \sin(x)y = 2\cos^{8}(x^{8})\ \sin x(x) - 1\ y\left[\frac{\pi}{4}\right] = 3\sqrt{2}$

$0 \leq x \leq \frac{\pi}{2}$

sol:

first rewriting differentibal equation, to get derivative of a one.

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\ \sin(x) - \frac{1}{\cos(x)}$$

$$y' + \tan(x)y = 2\cos^2(x)\sin(x) - \sec(x).$$

Integerating factor:

$$u(t) = e^{\int \tan(x)dx} = e^{\ln|\sec x|} = e^{\ln \sec(x)} = \sec(x)$$

Now,

$$\int \tan(x)dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1} = \ln|\sec(x)|$$

also,

$$e^{\ln f(x)} = f(x).$$

xing integeral on b.s

$\sec(x)y' + \sec(x) + \tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x).$

$$= [\sec(x)(y)]' = 2\cos(x)\sin(x) - \sec^2(x)$$

integerating on b.s

$$\int \sec(x)y(x))'dx = \int 2\cos(x)\sin(x) - \sec^2(x)dx$$

$$\sec(x)\ y(x) = \int \sin(2x) - \sec^2(x)dx$$

$$\sec(x)y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + c$$

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Note that trig formula $\sin(2\theta) = 2\sin\theta\cos\theta$.

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x).$$

$$= -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x).$$

Now, find value of c.

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right)$$

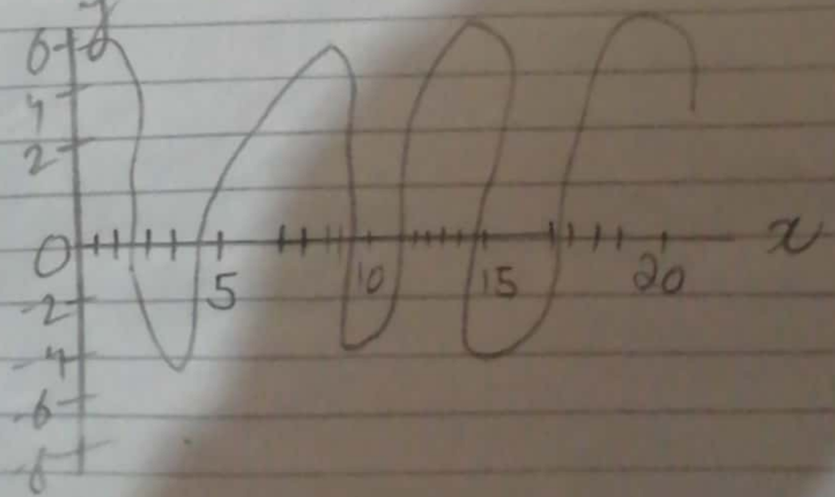
$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2}$$

$$c = 7.$$

Solution is then.

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x).$$

Graph is.



(iii) $x' + 2x = \sin(t)$

Sol:

first-order linear differential equation

$x'(t) = \sin(t) - 2x(t)$.

$x'(t) + 2x(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{t}$

let $\mu(t) = e^{\int 2dt} = e^{2t}$

Multiply b.s by $\mu(t)$.

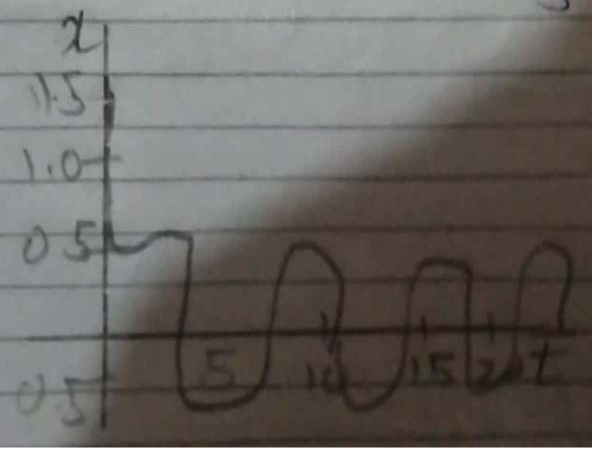
$e^{2t}\frac{dx(t)}{dt} + 2(e^{2t})x(t) = e^{2t}\sin(t)$.

subsitute $2e^{2t} = \frac{d}{dt}(e^{2t})$.

$e^{2t}\frac{dx(t)}{dt} + \frac{d}{dt}(e^{2t})x(t) = e^{2t}\sin(t)$.

solution

$x(t) = c_1 e^{-2t} + \frac{2\sin(t)}{5} - \frac{\cos(t)}{5}$.



Q3) Solve the following IVP for exact equation, and find interval of validity for solution.

① $2xy - 9x^2 + (2y + x^2+1)\frac{dy}{dx} = 0, \ y(0) = -3.$

Sol.
first, make M and N for exact equation

$$M = 2xy - 9x^2, \quad M_y = 2x$$
$$N = 2y + x^2+1, \quad N_x = 2x$$

Now, $\Psi(x,y)$.

$$\Psi_x = M$$
$$\Psi = N$$

Now integrating $\Psi = \int M dx$ or $\Psi = \int N dy$

So,

$$\Psi(x,y) = \int 2xy \ 9x^2 dx = x^2y - 3x^2 + h(y)$$
$$\Psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$$
$$h'(y) = 2y+1$$
$$h(y) = \int 2y + 1 dy = y^2 + y + k$$
$$\Psi(x,y) = x^2y - 3x^2 + y^2 + y + k = y^2 + (x^2+1)y - 3x^3$$

k

$$y^2 + (x^2+1)y - 3x^3 + k = c$$
$$y^2 + (x^2+1)y - 3x^3 = c - k$$
$$y^2 + (x^2+1)y = 3x^3 = c$$

$y^2 + (x^2+1)y - 3x^3 = c - k.$

$y^2 + (x^2+1)y - 3x^3 = c.$

$(-3)^2 + (0+1)(-3) - 3(0)^3 = c \Rightarrow c = 6.$

then

$y^2 + (x^2+1)y - 3x^3 - 6 = 0.$

$$y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^3-6)}}{2(1)}.$$

$$= \frac{(-x^2+1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}.$$

$$-3 = y(0) = \frac{-1 + \sqrt{25}}{2}.$$

$$= \frac{-1 + 5}{2} \quad , -3, 2$$

$$y(x) = \frac{-(x^2+1) - \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}.$$

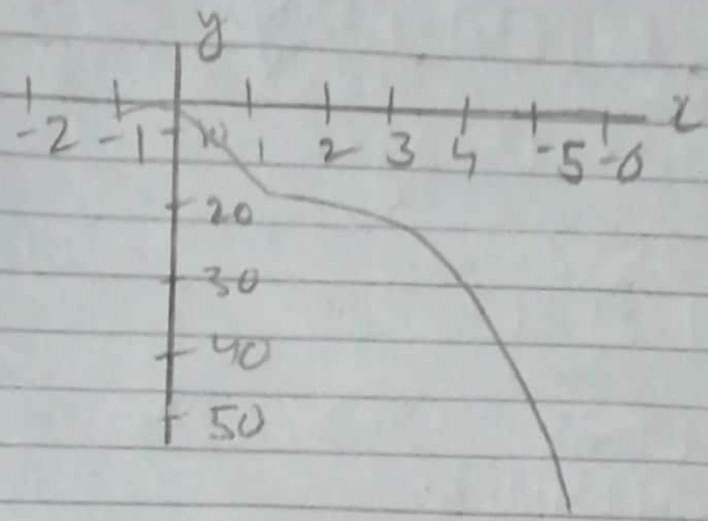
$x^4 + 12x^3 + 2x^2 + 25 = 0.$

Now,

$-\infty < x \leq -11.81$

$-1.396 \leq x < \infty.$

so, $-1.369113 \leq \infty.$



(ii) $\frac{2ty}{t^2+1} - 2t - (2-\ln(t^2+1))y' = 0, \quad y(5) = 0.$

Sol:
first separting it.

$$\frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)y' = 0$$

Now, M and N,

$$M = \frac{2ty}{t^2+1} - 2t, \quad M_y = \frac{2t}{t^2+1}$$

$$N = \ln(t^2+1) - 2 \quad N_t = \frac{2t}{t^2+1}$$

Integerate first case

$$\Psi(t,y) = \int \frac{2ty}{t^2+1} - 2t \, dt = y\ln(t^2+1) - t^2 + h(y).$$

differentitate y w.r.t N.

$$\Psi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2 = N$$

we got,
$h'(y) = -2, \quad h(y) = -2y$

$\Psi(t,y) = y\ln(t^2+1) - t^2 - 2y$

$$y\ln(t^2+1) - t^2 - 2y = c$$

$$-25 = c$$

$$y(\ln(t^2+1) - 2) - t^2 = -25$$

$$y(t) = \frac{t^2-25}{\ln(t^2+1)-2}$$

$$\ln(t^2+1) - 2 = 0$$
$$\ln(t^2+1) = 2$$
$$t^2+1 = e^2$$
$$t = \pm\sqrt{e^2-1}$$

$$-\infty < t < -\sqrt{e^2-1}$$
$$-\sqrt{e^2-1} < t < \sqrt{e^2-1}$$
$$\sqrt{e^2-1} < t < \infty$$

$t=5$, and $\sqrt{e^2-1} < t < \infty$