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**Program:** BS Software Engineering

**Course Title:** Differential Equations

Q 1: a) Define differential equation along with 2 examples? (1+1 Marks)

## Answer:

A **differential equation** is an **equation** which contains one or more terms which involve the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable) dy/dx = f(x) Here “x” is an independent variable and “y” is a dependent variable. For **example**, dy/dx = 5x

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b) Define a Separable Differential Equation (DE)? (1+4+3 Marks)

## Answer:

For a **differential equation** to be **separable**, the variables must be able to be separated. This means that the **equation** can be rearranged so that all terms containing one of the variables are on one side of the equal sign, while all terms containing the other variable are on the other side.

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i. Solve the following Initial Value Problem (IVP) using separable DE and find the interval of validity of the solution.

(a) $\boldsymbol{y'} = \boldsymbol{xy3}\ \sqrt{\ }\mathbf{1} + \boldsymbol{x}\ \mathbf{2}\ (\mathbf{0}) = \mathbf{-1}$

## Solution:

Q1 (1) Part (a) $y' = \frac{xy^3}{\sqrt{1+x^2}}$ $y(0) = -1$

Solution:- First separate and then integrate both sides.

$$y^{-3}\,dy = x(1+x^2)^{-1/2}\,dx$$

$$\int y^{-3}\,dy = \int x(1+x^2)^{-1/2}\,dx$$

$$\frac{-1}{2y^2} = \sqrt{1+x^2} + c$$

Apply the initial condition to get the value of c

$$\frac{-1}{2} = \sqrt{1} + c$$

$$c = -3/2$$

The implicit solution is then

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - 3/2$$

Now let's solve for $y(x)$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y(x) = \pm\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Lets get the interval of validity.

(b) $y' = e^{-y}(2x - 4)$ $(5) = 0$

Sol:

$dy/dx=e^{-y}(2x-4)$

$y(5)=0$

by multiplying by $e^y$ and by dx,

$\Rightarrow e^y dy=(2x-4)dx$

by integrating,

$\Rightarrow \int e^y dy=\int(2x-4)dx$

$\Rightarrow e^y=x^2-4x+C$

by taking the natural log,

$\Rightarrow y=\ln(x^2-4x+C)$

Now, we need to find C using y(5)=0.

$y(5)=\ln((5)^2-4(5)+C)=\ln(5+C)=0$

$\Rightarrow 5+C=1 \Rightarrow C=-4$

Hence, the solution is $y=\ln(x^2-4x-4)$.

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Q 2: a) Solve the following IVP using Linear Differential method (2+5+3 Marks)

(i) Explain the steps for solving Linear Differential Equation.

## Answer:

**Here is a step-by-step method for solving them:**

1. Substitute y = uv, and. ...
2. Factor the parts involving v.

3. Put the v term equal to zero (this gives a **differential equation** in u and x which can be **solved** in the next **step**)
4. **Solve** using separation of variables to find u.
5. Substitute u back into the **equation** we got at **step** 2

(ii) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1 \quad y\left[\frac{\pi}{4}\right] = 3\sqrt{2},\ 0 \le x \le \frac{\pi}{2}$

(iii) $x' + 2x = \sin$�

**Solution:**

(ii) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$

$y\left[\frac{\pi}{4}\right] = 3\sqrt{2} \quad 0 \leq x \leq \frac{\pi}{2}$

Solution

$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$

$y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$y' + \frac{\sin(x)}{\cos(x)} y = 2\cos^2(x)\sin(x) - \frac{1}{\cos x}$

$y' + \tan(x)y = 2\cos^2(x)\sin x - \sec(x)$

$u(t) = e^{\int \tan(x)dx} = e^{\ln|\sec(x)|} = e^{\ln\sec(x)} = \sec(x)$

$\int \tan(x)dx = -\ln|\cos(x)| = \ln|\cos(x)| = -\ln|\sec(x)|$

$\sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$

$\int (\sec(x)y(x))dx = \int 2\cos(x)\sin(x) - \sec^2(x)\,dx$

$\sec(x)y(x) = \int \sin(2x) - \sec^2(x)\,dx$

$\sec(x)y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + c$

$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + \frac{c}{\cos(x)}$

$= -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos x$

put the value of y & x

$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + \cos$

$3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\ \frac{\sqrt{2}}{2}$

$c = 7$

$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$

(iii) $x' + 2x = \sin t$

Solution

$x' + P(x)x = Q(x)$

$P(x) = 2$

$Q(x) = \sin t$

$I(x) = e^{\int P(x)dx} = e^{\int 2dt}$

$I(x) = e^{2t}$

$I(x) = 2t$

Now,

$x = \frac{1}{I(x}\left[\int I(x) \cdot Q(x)\,dt + c\right]$

$x = \frac{1}{2t}\left(\int 2t \cdot \sin t\,dt + c\right)$

$x = \frac{1}{2t}\left(2\int \frac{1}{2}(1 - \cos(2t)\,dt + c\right)$

$x = \frac{1}{2t}\left(\int 1 - \cos(2t)\,dt + c\right)$

$$x = \frac{1}{2t}\left(t - \frac{1}{2}\sin(2t) + c\right)$$

$$x = \frac{t}{2t} - \frac{1}{4t}\sin(2t) + \frac{c}{2t}$$

Q 3: Solve the following IVP for the exact equation and find the interval of validity for the solution. (5+5 Marks)

(i) $2xy - 9x\ 2 + (2y + x\ 2 + 1)\ dy\ dx = 0, \quad y(0) = -3$

(ii) $2ty\ t\ 2{+}1 - 2t - (2 - ln(t\ 2 + 1))y' = 0 \quad y(5) = 0$

Question 3rd Part (1)

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$

Solution $y(0) = -3$

$M = 2xy - 9x^2 \qquad M_y = 2x$

$N = 2y + x^2 \qquad N_x = 2x$

Now, how do we actually find $\psi(x,y)$

$\psi = M \ , \ \psi = N$

$\psi = \int M dx$ or $\psi = \int N dy$

$\psi_y = x^2 + h(y) = 2y + x^2 + 1 = N$

$h'(y) = 2y + 1$

$h(y) = \int 2y + 1 dy = y^2 + y + K$

$\psi(x,y) = x^2 y - 3x^3 + y^2 + y + K =$

$y^2 + (x^2+1)y - 3x^2 + K$

$y^2 + (x^2+1)y + 3x^2 + K = C$

$y^2 + (x^2+1)y - 3x^2 = C - K$

$y^2 + (x^2+1)y - 3x^3 = c$

Initial condition to find c

$(-3)^2 + (0+1)(-3) - 3(0)^3 = c \Longrightarrow c = 6$

Put the value of c

$y^2 + (x^2+1)y - 3x^2 - 6 = 0$

Quadratic formula

$y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^3-6)}}{2(1)}$

$\frac{-(x^2+1) \mp \sqrt{x^4+12x^3+2x^2+25}}{2}$

$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} \to -3, 2$

$y(x) = \frac{-(x^2+1) - \sqrt{x^4+12x^2+25}}{2}$

$y(x) = x^4 + 12x^3 + 2x^2 + 25 = 0$.

Q3 Part 2nd.

$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0 \quad y(5) = 0$

Solution

$M = \frac{2ty}{t^2+1} - 2t \quad M_y = \frac{2t}{t^2+1}$

$N = \ln(t^2+1) - 2 \quad N_t = \frac{2t}{t^2+1}$

integrate the first one.

$$\psi(x,y) = \int \frac{2ty}{t^2+1} - 2t \, dy = y \ln(t^2+1) - t^2 + h(y)$$

Now differentiate

$$\psi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2 = N$$

$$h'(y) = -2 \Rightarrow h(y) = -2y$$

$$\psi(t,y) = y \ln(t^2+1) - t^2 - 2y$$

$$y \ln(t^2+1) - t^2 - 2y = c$$

$$c = -25$$

$$y(\ln(t^2+1) - 2) - t^2 = -25$$

$$y(t) = \frac{t^2 - 25}{\ln(t^2+1) - 2}$$

$$\ln(t^2+1) - 2 = 0$$

$$\ln(t^2+1) = 2$$

$$t^2 + 1 = e^2$$

$$t = \pm\sqrt{e^2 - 1}$$ Answer.