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Subject Differential equation.

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Q# 1: Find general solution

$$4y'' - 20y' + 25y = 0$$

Sol: This is second order homogenous differential equation with constant coefficient.

$$ay'' + by' + cy = 0$$

and solution for this is

$$y = e^{\lambda} \quad \text{—(i)}$$

General solution

$$y = C_1 e^{\lambda} + C_2 + e^{\lambda}$$

Now

$$4 \frac{d^2}{dx^2}(y) - 20\frac{d}{dx}(y) + 25(-y) = 0 \quad \text{—(A)}$$

put eq (i) in eq (A)

$$\Rightarrow 4 \frac{d^2}{dx^2}(e^{\lambda x}) - 20\frac{d}{dx}(e^{\lambda x}) + 25e^{\lambda x} = 0$$

$$\Rightarrow \frac{d^2}{dx^2} e^{\lambda x} = \lambda^2 e^{\lambda x} \longrightarrow (B)$$

put eq (B) and eq (i) in eq (A)

$$\Rightarrow 4\lambda^2 e^{\lambda x} - 20 e^{\lambda x} + 25 e^{\lambda x} = 0$$

$$\Rightarrow e^{\lambda x}(4\lambda^2 - 20\lambda + 25) = 0$$

$$\Rightarrow e^{\lambda x} \neq 0$$

$$\Rightarrow 4x^2 - 20\lambda + 25 = 0$$

$$\Rightarrow (2\lambda - 5)^2 = 0$$

$$\lambda = 5/2 \qquad \lambda = 5/2$$

$$\Rightarrow y(x) = y_1(x) + y_2(x)$$

$$\Rightarrow \boxed{y(x) = C_1 e^{5/2 x} + C_2 x e^{5/2}}$$

Q2a: Calculate the initial value problem $y''+2y'+y=0$

$y(0)=4.\ y'(0)=-6.$

Sol:- $y''+2y'+y=0$

$\lambda^2+2\lambda+1=0$

$\lambda^2+\lambda+\lambda+1=0$

$\lambda(\lambda+1)+1(\lambda+1)=0$

$\lambda=-1\ ,\ \lambda=-1$

roots are real & equal

The multiply of root $\lambda=-1$ is 2 which gives $y_1(x)=C_1e^{-x}$

$y_2(x)=C_2e^{-x}$

as solution where C_1 are constant

The general solution is the sum of the above solution.

$y(x)=y_1(x)+y_2(x)=C_1e^{-x}+C_2e^{-x}x$

$y'=C_1e^{-x}+C_2e^{-x}$

$y'=C_1e^{-x}+C_2e^{-x}-xe^{-x}$

Solve for the unknown constant using the initial conditions
compute

$$\frac{dy(x)}{dx}$$

$$\frac{dy(x)}{dx} = \frac{d}{dx}\left(C_1 e^{-x} + C_2 e^{-x} x\right).$$

$$= C_1 e^{-x} + C_2 e^{-x} - C_2 e^{-x} x.$$

Now, $y = 4$, $x = 0$

Substitute $y = 4$ into $y(x) = e^{-x} C_1 + e^{-x} x C_2$

$$\boxed{C_1 = 4} \quad (1)$$

Now, $x = 0$, $y = 6$

Substitute $y'(0) = -6$

$$-6 = C_1 e^{-x} + C_2 e^{-x} - C_2 e^{-x} x$$

$$-6 = C_1 + C_2 \quad (2)$$

Adding (1) and (2)

$$\begin{aligned} 4 &= C_1 \\ -6 &= C_1 + C_2 \\ \hline -2 &= C_2 \end{aligned}$$

$$\boxed{C_1 = 4}$$

$$\boxed{C_2 = -2}$$

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Subsitute $C_1 = 4$ & $C_2 = -2$ into

$$y' = C_1 e^{-x} + C_2 e^{-x} x.$$

$$\boxed{y' = -2e^{-x}(x-2)}$$

Q # 2(b) :- Analyze the general solution of

$$x^2 y'' + 3xy' + y = 0$$

Sol:- $a = 3 \quad , \quad b = 1$

$$m^2 + (a-1)m + b = 0$$

$$m^2 + (3-1)m + 1 = 0$$

$$m^2 + 2m + 1 = 0$$

$$m^2 + m + m + 1 = 0$$

$$m(m+1) + 1(m+1) = 0$$

$$(m+1)(m+1) = 0$$

$$m = -1 \quad , \quad m = -1$$

roots are real and equal

so,

$$y = (C_1 + C_2 \ln x)\, x^m$$

Q#3: Examine the method of undetermined coefficient method for

$$y'' + y' - 6y = 6x^3 - 3x^2 + 12x.$$

Sol: $y'' + y' - 6y = 6x^3 - 3x^2 + 12x$ —— (1)

for Homogenous equation

$a = 1$, $b = -6$

$\lambda^2 + \lambda - 6$

$\lambda^2 + 3\lambda - 2\lambda - 6.$

$\lambda(\lambda + 3) - 2(\lambda + 3)$

$\lambda - 2 = 0$, $\lambda + 3 = 0$

$\lambda = 2$, $\lambda = -3$

So,

$y = C_1 e^{2x} + C_2 e^{-3x}$, choice for y_p.

$y_p = K_3 x^3 + K_2 x^2 + K_1 x^1 + K_0.$

$y_0 = 3K_3 x^3 + 2K_2 x' + K_1$

$y_p = 6K_3 x + 2x_2$

put in eq (1)

$$\Rightarrow \underbrace{\cancel{6K_3}x + 2x}_{y''} + \underline{3K_3x^2 + 2K_2x + x_1}$$

$$-6K_3x^3 - 6x_2x^2 - 6K_1x - 6K_0$$

$$= 6K_3 \cancel{x^3} - 3K_2x^2 + 12x$$

$$-6k_3x^3 + (3k_3 - 6k_2)x^2 + (6k_3 + 2k_2 - 6k_1)x$$
$$+ 2k_2 + k_1 - 6k_0 = 6x^3 - 3x^2 + 12x$$

$$-6k_3 = 6$$

$$\Rightarrow \quad \boxed{k_3 = -1}$$

$$3k_3 - 6k_2 = -3$$
$$3(-1) - 6k_2 = -3$$
$$-k_2 = 0 \Rightarrow \boxed{k_2 = 0}$$

$$6k_3 + 2k_2 - 6k_1 = 12$$
$$6(-1) + 1(0) - 6k_1 = 12$$
$$-6 + 0 - 6k_1 = 12$$
$$k_1 = -18/6$$
$$\boxed{k_1 = -3}$$

$$-2k_1 + k_1 + k_0 = 0$$
$$-2(-3) - 2 + 2k_2 = 0$$
$$\boxed{k_0 = -1/2}$$

So, $y_p = -x^3 + 0x^2 - 3x - \frac{1}{2} = -x^3 - 3x - \frac{1}{2}$

Q # 4: Examine the method of variation parameters for

$$y'' - 4y' + 4y = x^2 e^{2x}$$

Sol: $y'' - 4y' + 4y = x^2 e^{2x}$

for equation

$$y'' - 4y' + 4y = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda^2 - 2\lambda - 2\lambda + 4 = 0$$

$$\lambda(\lambda - 2) - 2(\lambda - 2) = 0$$

$$(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 2, \; \lambda = 2$$

Roots are real and equal

$$y = (c_1 + c_2 x) e^{2x}$$

$$y_1 = c_1 e^{2x} + c_2 x e^{2x}$$

$$y_1 = c_1 e^{2x} + c_2 x e^{2x}$$

$$y_1 = e^{2x}, \; y_2 = x e^{2x}$$

$$y_1' = 2e^{2x}, \; y_2' = e^{2x} + 2e^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2ex^2e \end{vmatrix}$$

$$W = e^{4x} + 2\cancel{xe^{4}} - 2x\cancel{e^{4x}}$$

$$w = e^{4x}$$

$$y_p = -y_1 \int \frac{y_2 r(x)}{w} + y_2 \int \frac{y_1 y(x)}{w}$$

$$y_p = e^{2x} \int \frac{xe^{3x} \cdot x^2 e^{2x}}{e^{4x}} dx + xe^{3x} \int \frac{e^{2x} \cdot x^2 e^{2x} dx}{e^{4x}}$$

$$y_p = -e^{2x} \int x^3 \frac{\cancel{e^{4x}}}{\cancel{e^{4x}}} + xe^{2x} \int x^3 \frac{\cancel{e^{4x}}}{\cancel{e^{4x}}} dx$$

$$y_p = -e^{2x} \int x^2 dx + xe^{2x} \int x^2 dx$$

$$y_p = -e^{3x} \cdot \frac{x^4}{4} + xe^{3x} \cdot \frac{x^3}{3}$$

so, $y = y_h + y_p$

$$y = C_1 e^{2x} + C_2 x e^{2x} - e^{2x} \frac{x^4}{4} + xe^{2x} \frac{x^3}{3}$$

Q#5: Identify an ODE $y''+ay'+by=0$ for base $1, e^{-3x}$.

Sol: e^{-3x} will e^{3x}

$\Rightarrow y_1 = e^{0x}, \ y_2 = e^{3x}$

$\Rightarrow y = C_1 e^{0x} + C_2 e^{3x}$

So, roots are real and distinct.

$$y = (C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x})$$

So, $\lambda_1 = 0, \ \lambda_2 = 3$

$\lambda_1 = 0, \ \lambda_2 - 3 = 0$

$(\lambda)(\lambda - 3) = 0$

$\lambda^2 - 3\lambda = 0$

So,

$\lambda^2 - a\lambda + b = 0$

As $a = -3, \ b = 0$

So,

$$y'' + ay' + by = 0$$

$$\boxed{y'' - 3y' = 0}$$