University: Iqra National University.

Name: Alamgir khan

Teacher name: Madam: Shomaila Mazhar

I D number: 7282

Subject: Differential Equations.

Depeoment: Civil engeering.

Semister: 16^{th}

Date: 30, June, 2020

Question number (ONE)(01)

Q1: The wave Equation:.

we generally visit beach and if we stand on an ocean Shore and take a snapshots of waves. .... the one dimensional wave equation.

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

where w is the wave height.... .... poopagated.

Show the following function..... ..... debivation.

i) $w = \sin(x+ct) + \cos(2x+2ct)$

ii) $w = \tan(2x+ct)$

Solution:-

(i) $w = \sin(x+ct) + \cos(2x+2ct)$

Given. $\frac{\partial^2 w}{\partial t^2} = c' \frac{\partial^2 w}{\partial x^2} \rightarrow (1)$

Scanned with CamScanner

Now: $\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right)$

$\Rightarrow \frac{\partial}{\partial t}(\sin(x+ct)) + \frac{\partial}{\partial t}(\cos(2x+2ct))$

$\Rightarrow \frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$

Now: $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right)$

$\Rightarrow \boxed{\frac{\partial^2 w}{\partial t} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct}$

Now $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$

$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$

4 $\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right)$

$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$

Scanned with CamScanner

(1) $\Rightarrow$ -

$- \varepsilon \sin(\gamma + ct) - 4c^2 \cos(2x + 2ct) =$

$= c^2 [- \sin \gamma + ct) - 4 \cos(x + ct)]$

$= - \varepsilon \sin(\gamma + ct) - 4c^2 \cos(\cancel{2}x + \cancel{2}ct =$

$- c \sin(\cancel{x} + ct) - 4c^2 \cancel{c}os\ x + ct)$

$\Rightarrow \boxed{0 = 0 \text{ (Satisfied)}}$

$\Rightarrow$ (iii) $w = \tan(2x + ct)$.

Now $\dfrac{\partial w}{\partial t} = c \sec^2(2x + ct)$

and $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial t}(c \sec^2(2x + ct)$

$= c^2 \cdot 2 \sec^2(2x + ct) \tan(2x + ct)$

Now:

$\dfrac{\partial w}{\partial x} = 2 \sec^2(2x + ct)$

$\dfrac{\partial^2 w}{\partial z^2} = 4 \sec^2(2x + ct) \tan(2x + ct)$

(1) =>

$$4c^2 \sec^2(2x+ct)\tan(2x+ct) =$$

$$4c^2 \sec^2(\cancel{2x+c}t)\tan(2x+ct)$$

=> $\boxed{= \quad 0 = 0 \quad \text{Satisfied}}$

(७)

Question number (Two) (02)

Q2: Expand the following in a Fourier series.

$F(x) = x, \quad -\pi < x \leq 0$

$= 2x, \quad 0 \leq x \leq \pi.$

Solution:

Given: $F(x) \begin{cases} x, & -\pi < x \leq 0 \\ 2x, & 0 \leq x \leq \pi \end{cases}$ — (1)

Let $F(x) = \frac{a_0}{2} + \sum_{x=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Now $a_0 = \frac{1}{\pi} \int F(x)\, dx = \frac{1}{\pi} \left\{ \int F(x)\, dx + \int^{\pi} F(x)\, dx \right\}.$

$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} x\, dx + \int_{0} 2x\, dx \right\} = \frac{1}{\pi} \left\{ \frac{x^2}{2} \Big|^{0} + x^2 \Big|_{0}^{\pi} \right\}$

$= \frac{1}{\pi} \left\{ \left(\frac{0^2}{2} - \frac{\pi}{2} \right)^2 + (\pi^2 - 0) \right\} = \frac{1}{\pi} \left[-\frac{\pi^2}{2} + \pi^2 \right\}$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2}\right\} = \frac{\pi}{2}$$

$\Rightarrow$ Now

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi} 2x\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left|x\frac{\sin nx}{n}\right|_{-\pi}^{0} - \int^{0}\frac{\sin nx}{x}\,dx\right\}$$

$$+ \frac{2}{\pi}\left\{\left|\frac{x\sin nx}{n}\right|_{0}^{\pi} - \int_{0}^{\pi}\frac{\sin nx}{n}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi\sin n(-\pi)}{n} - \frac{1}{n}\left|\frac{(-\cos nx}{n}\right|_{-\pi}^{0}\right\}$$

$$+ \frac{2}{\pi}\left\{0 - \frac{1}{n}\left|\frac{(-\cos x)}{n}\right|_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{0 + \frac{1}{n^2}\left(\cos n0 - \cos n(-\pi)\right)\right\}$$

$$+ \frac{2}{2n\pi}\left\{\cos n\pi\} - \cos n0\right\}$$

$$= \frac{1}{n^2\pi}(1-\cos n\pi) + \frac{2}{n^2\pi}(\cos n\pi - 1)$$

$$= \left(-\frac{1}{n^2\pi} + \frac{2}{n^2\pi}\cos n\pi + \frac{1}{n^2\pi} - \frac{2}{n^2\pi}\right.$$

$$= \frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}$$

$$\Rightarrow \boxed{a_n = \frac{\cos n\pi - 1}{n^2\pi}}$$

$\Rightarrow$ Now: $b_n = \frac{1}{\pi}\int^{\pi} F(x)\sin nx\,dx.$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{1}{\pi}\int^{\pi} 2x\sin nx\,dx.$$

$$= \frac{1}{\pi}\left\{\left[\left| x\left(\frac{-\cos nx}{n}\right)\right|_{-\pi}^{0} - \int_{-\pi}^{0} 1\left(\frac{-\cos nx}{n}\right)dx\right]\right\}$$

$$+\frac{2}{\pi}\left\{\left|x\left(-\frac{\cos nx}{n}\right)\right|_0^{\pi}-\int_0^{\pi}-\frac{\cos nx}{n}\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\frac{1}{n}\left(0-(-\pi)\right)(-\cos n\pi)+\frac{1}{n}\left|\frac{\sin nx}{n}\right|_{\pi}^{0}\right.$$

$$\left.+\frac{2}{\pi}\left\{-\frac{\pi\cos n\pi}{n}+\frac{1}{n}\left|\frac{\sin nx}{n}\right|_0^{\pi}\right\}\right.$$

$$=\frac{1}{\pi}\left\{-\frac{\pi\cos n\pi}{n}+\frac{1}{n^2}(0-0)\right\}$$

$$+\frac{2}{\pi}\left\{-\frac{\pi\cos n\pi}{n}+\frac{1}{n^2}(0-0)\right\}$$

$$=-\frac{\cos n\pi}{n}-\frac{2\cos n\pi}{n}$$

$$b_n=-\frac{3\cos n\pi}{n}$$

Hence.

$$\Rightarrow\quad F(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{\cos n\pi-1}{n^2\pi}\cos nx+\left(-\frac{3\cos n\pi}{n}\sin nx\right)\right).$$

Question number (Three) (03)

Q3: Solve the initial value Problem.

$$y'' - 4y' + 13y = 8\sin 3x.$$

$$y(0) = 1 \text{ and } y'(0) = 2$$

Solution:-

$$y'' - 4y' + 13y = 8\sin 3x$$

we have to find $y = y_c = y_p$

=> For y_c the characteristic (auxiliary Eqn) Eqn is:

$$m^2 - 4m + 13 = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16-52}}{2} \Rightarrow m = \frac{4 \pm 6i}{2}$$

$$\Rightarrow m = 2 \pm 3i \; ; \; \alpha = 2 \; \& \; \beta = 3$$

So $y_c = e^{2x}\{c_1\cos 3x + c_2\sin 3x\}$

For y_p let

$$\Rightarrow = y_p = \text{Imag}\left(\frac{1}{m^2-4m+13}\, 8e^{3ix}\right).$$

$$= 8\ \text{Imag}\ \frac{e^{3jx}}{(3j)^2-4(3j)+13}$$

$$= 8\ \text{Imag}\ \frac{3jx}{-9-12j+13}$$

$$= 8\ \text{Imag}\ \frac{e^{3jx}}{4-12j}$$

$$y_p = 2\ \text{Imag}\ \frac{e^{3jx}}{(1-3j)} \times \frac{(1+3j)}{(1+3j)}$$

$$y_p = 2\ \text{Imag}\ \frac{(1+3j\,(e^{3jx})}{(1)^2-(3j)^2}$$

$$y_p = 2\ \text{Imag}\left(\frac{(1+3j)(e^{3jx})}{10}\right)$$

$$y_p = \frac{2}{10}\left(\text{Imag}\ (1+3j)(\cos 3x + j\sin 3x)\right)$$

$$y_p = \frac{2}{10}\left[\sin 3x + 3\cos 3x\right)$$

So the general solution is.

$$y = y_c + y_p.$$

$\Rightarrow y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x + \frac{2}{10} [\sin 3x + 3\cos 3x)$

= Now use the initial condition.

$y(0) = 1$

$= y(0) = C_1 e^{(0)} \cos(0) + C_2 e^{(0)} \sin(0) + \frac{2}{10} (\sin(0) + 3\cos(0))$

$= 1 = C_1 (1) + 0 + 0 + \frac{2}{10} (3(1)).$

$= 1 = C_1 + \frac{6}{10} \Rightarrow \boxed{C_1 = 1 - \frac{6}{10} = \frac{4}{10} = 2/5}$

$\Rightarrow$ Again use the another initial condition:

$y'(0) = 2$

So, $y' = C_1 2e^{2x} \cos 3x + C_1 e^{2x} (-3\sin 3x) + C_2 2e^{2x} \sin 3x + C_2 e^{2x} (3\cos 3x) + \frac{2}{10} (\cos 3x - 3\sin 3x)$

$y'(0) = C_1 2e^{(0)} \cos(0) + C_1 e^{(0)} (-3\sin 0) + C_2 2e^{(0)} \sin(0) + C_2 e^{(0)} (3\cos \cos)$

$+ \frac{2}{10}(\cos(0) - 3(\sin(0))$

$2 = 2c_1 + 0 + 0 + c_2 \, 3(1) + \frac{2}{10}(1 - 3(0)) \, dx$

$2 = 2c_1 + 3c_2 + \frac{2}{10}$

$\Rightarrow$ $\boxed{\text{use } c_1 = 2/5}$

$\frac{1}{3}\left(2 - \frac{4}{5} - \frac{2}{10}\right) = c_2 \Rightarrow$

$\Rightarrow \boxed{c_2 = \frac{1}{3} \frac{(20 - 8 - 2)}{10} = 1/3}$

The General Solution is

$y = 2/5 \, e^{2x} \cos 3x + 1/3 \, e^{2x} \sin 3x + \frac{2}{10}\left[\sin 3x + 3\cos 3x\right].$

- b The required Solution -

Question number (four) (04)

Q4: Solve?

$(D^2 - DD')z = \cos x \cos 2y.$

Solution:

$(D^2 - DD')z = \cos x \cos 2y.$

The auxilary equation is

$m^2 - m = 0 \Rightarrow m = 0, m = 1.$

$\Rightarrow$ Hence the complementary function is given by:

$z_c = f_1(y) + f_2(y+x)$

for the particular Integral, we have $z_P = \frac{1}{D^2 - DD'} \cos x \cos 2y.$

$\Rightarrow = \frac{1}{2} \cdot \frac{1}{D^2 - DD'}[\cos(x-2y) + \cos(x-2y)]$

$\Rightarrow = \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x+2y) + \frac{1}{D^2 DD'}\cos \ldots\right.$

$\left.(x-2y)\right]$

$$= \frac{1}{2}\left[\frac{1}{-1+2}\cos(x+2y) + \frac{1}{-1-2}\cos(x-2y)\right]$$

$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

⇒ Hence the complete solution is given by:

Answer $Z = f_1(z) + f_2(y+x) + \frac{1}{2}\cos(x+2y) \cdots$

$- \frac{1}{6}\cos(x\ 2y)$.

Answer.