

Q1
Adjoint:

i) $A = \begin{bmatrix} 1 & 2 & 2nd-ID \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Sol:-

As: $A = \begin{bmatrix} 1 & 2 & 2nd-ID \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

put: $ID = 5$

\therefore we get; $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Now: Matrix A adjoint:

Adjoint Square MC

A is $(\text{cofactors}(A))^T$.

\therefore Minors and cofactors for; $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$:

* Minors :- A minor M_{ij} is the determinant of the square matrix formed by deleting Row i and Column j from larger square matrix A .

* Co-factors :- A co-factor C_{ij} is \bullet Minor $+M_{ij}$ when $i+j$ is even, and \bullet Minor $-M_{ij}$ when $i+j$ is odd, thus; $C_{ij} = (-1)^{i+j} M_{ij}$

i.e.

$$M_{1,1} : \det \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = 6 - 1 = \boxed{5}$$

$$M_{1,2} : \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = 4 - 3 = \boxed{1}$$

$$M_{1,3} : \det \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} = 2 - 9 = \boxed{-7}$$

$$M_{2,1} : \det \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix} = 4 - 5 = \boxed{-1}$$

$$M_{2,2} : \det \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix} = 2 - 15 = \boxed{-13}$$

$$M_{2,3} : \det \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = 1 - 6 = \boxed{-5}$$

$$M_{3,1} : \det \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} = 2 - 15 = \boxed{-13}$$

$$M_{3,2} : \det \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix} = 1 - 10 = \boxed{-9}$$

$$M_{3,3} : \det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = 3 - 4 = \boxed{-1}$$

\therefore Matrix of Minors M_{ij}

$$\text{is } \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{pmatrix} = \begin{bmatrix} 5 & 1 & -7 \\ -1 & -13 & -5 \\ -13 & -9 & -1 \end{bmatrix}$$

Also; Matrix of co-factors;

$$C_{i,j} = (-1)^{i+j} \cdot M_{i,j}$$

$$\therefore \begin{pmatrix} (-1)^{1+1} \cdot 5 & (-1)^{1+2} \cdot 1 & (-1)^{1+3} \cdot (-7) \\ (-1)^{2+1} \cdot (-1) & (-1)^{2+2} \cdot (-13) & (-1)^{2+3} \cdot (-5) \\ (-1)^{3+1} \cdot (-13) & (-1)^{3+2} \cdot (-9) & (-1)^{3+3} \cdot (-1) \end{pmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -7 \\ 1 & -13 & 5 \\ -13 & 9 & -1 \end{bmatrix}$$

Now;

Take Transpose of Matrix of Co-factors

i.e.

$$\begin{bmatrix} 5 & -1 & -7 \\ 1 & -13 & 5 \\ -13 & 9 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & 1 & -13 \\ -1 & -13 & 9 \\ -7 & 5 & -1 \end{bmatrix}$$

So;

adjoint of matrix A is;

$$\text{adj } A = \begin{bmatrix} 5 & 1 & -13 \\ -1 & -13 & 9 \\ -7 & 5 & -1 \end{bmatrix}$$

Result

$$(ii) \quad B = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 8 \end{bmatrix}$$

Solⁿ:

As; $B = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 8 \end{bmatrix}$

$$; \text{Adj } B \Rightarrow (\text{co-factors}(B))^T$$

\therefore Finding minors and cofactors;

i.e. $B = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 8 \end{bmatrix}$

$$M_{i,j} \text{ of } B = \begin{bmatrix} 8 & -24 & 1 \\ 42 & -1 & -26 \\ 37 & 14 & -11 \end{bmatrix}$$

Also;

$$C_{i,j} \text{ of } B = \begin{bmatrix} (+1)(8) & (-1)(-24) & (+1)(1) \\ (-1)(42) & (+1)(-1) & (-1)(-26) \\ (+1)(37) & (-1)(14) & (+1)(-11) \end{bmatrix}$$

$$\Rightarrow C_{i,j} \text{ of } B = \begin{bmatrix} 8 & 24 & 1 \\ -42 & -1 & 26 \\ 37 & -14 & -11 \end{bmatrix}$$

Take Transpose of $C_{i,j} \text{ of } B$.

$$\Rightarrow (C_{i,j} \text{ of } B)^T = \begin{bmatrix} 8 & -42 & 37 \\ 24 & -1 & -14 \\ 1 & 26 & -11 \end{bmatrix}^T$$

$$\Rightarrow \text{Adj } B = \begin{bmatrix} 8 & -42 & 37 \\ 24 & -1 & -14 \\ 1 & 26 & -11 \end{bmatrix} = \underline{\underline{\text{Result}_2}}$$

Q.2.5 - Co-factors of A_{21}, A_{31}, A_{33} .

i.e. $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

Sol:

As: $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

Finding Minors and then Co-factors;

$\therefore M_{i,j}(A) = \begin{bmatrix} 9 & -8 & -6 \\ 5 & -10 & 5 \\ -11 & 7 & -1 \end{bmatrix}$

$A_{21} \leftarrow$ (5) $\leftarrow A_{31}$ \leftarrow (-1) $\leftarrow A_{33}$

Now;

Co-factors of;

- $A_{21} = \left((-1)^{2+1} \cdot (5) \right) = (-1)^3 \cdot (5)$

$\Rightarrow \boxed{A_{21} = -5}$

- $A_{31} = \left((-1)^{3+1} \cdot (-11) \right) = (-1)^4 \cdot (-11)$

$\Rightarrow \boxed{A_{31} = -11}$

- $A_{33} = \left((-1)^{3+3} \cdot (-1) \right) = (-1)^6 \cdot (-1)$

$\Rightarrow \boxed{A_{33} = -1}$

Q.3:- Eigen values and Eigen Vectors;

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} \quad \& \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol:

As;

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix},$$

* Matrix Eigen vectors;

Let A be a square matrix, η a vector and λ a scalar that satisfy: $A\eta = \lambda\eta$, then η is an eigen vector of A and λ is the eigenvalue associated with it.

Now,

to find η eigen vectors, we need to solve $(A - \lambda I)\eta = 0$ for each eigen-value;

Eigen values of matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$ Eigen values;

Let A is a square matrix, η a vector and λ a scalar that satisfy $A\eta = \lambda\eta$, then λ is called the eigen values associated with the

eigen vector η of A .

The eigen values of A are the roots of the characteristic equation $\det(A - \lambda I) = 0$.

Now;

$$\det \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0 ?$$

So;

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1-0 & 1-0 \\ 1-0 & 3-\lambda & 2-0 \\ -1-0 & 1-0 & 2-\lambda \end{bmatrix}$$

o ~~2-0~~ ~~1-0~~ ~~1-0~~
o ~~1-0~~ ~~3-0~~ ~~2-0~~
o ~~-1-0~~ ~~1-0~~ ~~2-0~~

$$= \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 2 \\ -1 & 1 & 2-\lambda \end{bmatrix}$$

$$\therefore \det \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 2 \\ -1 & 1 & 2-\lambda \end{bmatrix} \quad \text{Expanding by}$$

Row - 1.

$$= (2-\lambda) \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} - (1) \begin{vmatrix} 1 & 2 \\ -1 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 3-\lambda \\ -1 & 1 \end{vmatrix}$$

$$= (2-\lambda) \left((3-\lambda)(2-\lambda) - 2 \right) - (1) \left((2-\lambda) - (-2) \right) + (1) \left(1 - (-1)(3-\lambda) \right)$$

$$= (2-\lambda) \left(6 - 3\lambda - 2\lambda + \lambda^2 - 2 \right) - (1) \left(2-\lambda + 2 \right) + (1) \left(1 + 3 - \lambda \right)$$

$$= (2-\lambda) \left(\lambda^2 - 5\lambda + 4 \right) - (1) \left(-\lambda + 4 \right) + 1 \left(-\lambda + 4 \right)$$

$$= \underline{2\lambda^2} - \underline{10\lambda} + 8 - \lambda^3 + \underline{5\lambda^2} - \underline{4\lambda} + \lambda - \cancel{4} - \cancel{\lambda} + \cancel{4}$$

$$= -\lambda^3 + 7\lambda^2 - 14\lambda + 8$$

Now;

$$-\lambda^3 + 7\lambda^2 - 14\lambda + 8 = 0$$

Solve, by factoring;

So, $-x^3 + 7x^2 - 14x + 8 = 0$

$\Rightarrow -1(x^3 - 7x^2 + 14x - 8) = 0 \rightarrow \textcircled{\#}$

we will use the "Rational Root Theorem"

to figure out factors;

i.e.:

$a_0 = 8, a_n = 1$

The divisors of a_0 : 1, 2, 4, 8

Also; divider of a_n : 1.

There, check following Rational numbers:

$\pm \frac{1, 2, 4, 8}{1}, \frac{1}{1}$, is a root of the expression, so factor out $(x-1)$

$= (x-1) \left(\frac{x^3 - 7x^2 + 14x - 8}{x-1} \right) \rightarrow \textcircled{\#}$

* $\frac{x^3 - 7x^2 + 14x - 8}{x-1}$, dividing leading coefficients of numerator and divisor

i.e. $\frac{x^3}{x} = x^2 \Rightarrow \text{Quotient} = x^2$

Now, Multiply $(x-1)$ by x^2

i.e: $x^2(x-1) = x^3 - x^2$.

Now, subtract, $(x^3 - x^2)$ from

$$x^3 - 7x^2 + 14x - 8$$

i.e: $(x^3 - 7x^2 + 14x - 8) - (x^3 - x^2)$

$$= \cancel{x^3} - 7x^2 + 14x - 8 - \cancel{x^3} + x^2$$

Remainder = $-6x^2 + 14x - 8$

$$\therefore \frac{x^3 - 7x^2 + 14x - 8}{x-1} = x^2 + \left(\frac{-6x^2 + 14x - 8}{x-1} \right)$$

↳ ①

Now;

$\frac{-6x^2 + 14x - 8}{x-1}$, divide the leading coefficients of numerator and the divisor,

i.e: $\frac{-6x^2}{x} = -6x$ → Quotient = $-6x$

Now, multiply; $-6x$ by $x-1$

i.e: $(-6x)(x-1) = -6x^2 + 6x$

Also: subtract $-6x^2 + 6x$ from $-6x^2 + 14x - 8$.

∴ i - e;

$$(-6x^2 + 14x - 8) - (-6x^2 + 6x)$$

$$= -6x^2 + 14x - 8 + 6x^2 - 6x$$

Remainder = $8x - 8$

Therefore;

$$\frac{-6x^2 + 14x - 8}{x - 1} = -6x + \frac{8x - 8}{x - 1} \rightarrow (2)$$

So, Refine eq 2) in eq 1)

$$\therefore \text{Eq 1)} \Rightarrow \frac{x^3 - 7x^2 + 14x - 8}{x - 1} = x^2 + (-6x) + \frac{8x - 8}{x - 1}$$

$$\Rightarrow \frac{x^3 - 7x^2 + 14x - 8}{x - 1} = x^2 - 6x + 8 \rightarrow (3)$$

Now; factorize " $x^2 - 6x + 8$ "

$$i - e: x^2 - 6x + 8 = x^2 - 2x - 4x + 8$$

$$\Rightarrow x^2 - 6x + 8 = x(x - 2) - 4(x - 2)$$

$$= (x - 2)(x - 4) \rightarrow (4)$$

Now, put eq 4) in eq *

$$\therefore \text{Eq 3)} \Rightarrow (x - 1) \left(\frac{x^3 - 7x^2 + 14x - 8}{x - 1} \right) = (x - 1)(x - 2)(x - 4)$$

*	x^2
	$- 2x$
	$- 4x$
	<hr style="width: 50%; margin: 0;"/>
	$- 6x$
	<hr style="width: 50%; margin: 0;"/>
	* $8x$
	* $- 4x$
	<hr style="width: 50%; margin: 0;"/>
	$+ 8x^2$
	<hr style="width: 50%; margin: 0;"/>

Now, put the above equation in eq #)

Ques: ∴ Eq #)

$$\Rightarrow -1(\lambda^3 - 7\lambda^2 + 14\lambda - 8) = -(\lambda - 1)(\lambda - 2)(\lambda - 4)$$

↳ 4)

As;

Eq # : $-1(\lambda^3 - 7\lambda^2 + 14\lambda - 8) = 0$

∴ Eq 4) $\Rightarrow -(\lambda - 1)(\lambda - 2)(\lambda - 4) = 0 \rightarrow 5)$

Now, using zero factor principle

i.e;

If $ab=0$, then $a=0$ OR $b=0$

OR (both $a=0$ and $b=0$).

Therefore;

$\lambda - 1 = 0$, $\lambda - 2 = 0$, $\lambda - 4 = 0$
 $\Rightarrow \boxed{\lambda = 1}$, $\Rightarrow \boxed{\lambda = 2}$, $\Rightarrow \boxed{\lambda = 4}$

So;

Eigenvalues are;

$\lambda = 1$, $\lambda = 2$, $\lambda = 4$.

Now;

Eigen vectors for : $\lambda = 1$:-

Solve for $(A - \lambda I)$:

i.e;

$A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ -1 & 1 & 2 \end{bmatrix} - \textcircled{1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (with $\lambda = 1$ indicated above)

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$$\Rightarrow A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \rightarrow \textcircled{a}$$

Now,

To solve; $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Reduce the matrix, eg)

i.e. $\begin{bmatrix} 1 & 1 & 1 \\ \textcircled{1} & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} a & - & b \\ 0 & \diagdown & \\ 0 & 0 & c \end{bmatrix}$, Reduce \uparrow matrix to Row Echelon form;

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ \textcircled{1} & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \text{ in } R_2 \text{ by performing } R_2$$

: $R_2 - (1) \cdot R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \text{ in } R_3 \text{ by performing } R_3$$

: $R_3 + (1) \cdot R_2$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$R_3 - \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

, cancel leading coefficient in R_3 by performing R_3 .

∴



Row Echelon form

of

$(A - \lambda I)$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now;

Reduce matrix to reduced Row Echelon form, i.e.:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{2} \cdot R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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cancel leading coefficient in R_1
by performing

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$$R_1 \leftarrow R_1 - 1R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, the system associated with the eigenvalue $\lambda=1$.

$$(A - \lambda I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

~~or~~

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= \begin{pmatrix} x + 0 + 0 \\ 0 + y + z \\ 0 + 0 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y + z \\ 0 \end{pmatrix},$$

This reduces to the following system of equations,

$$\begin{cases} \boxed{x = 0} \\ y + z = 0 \end{cases}$$

$$\Rightarrow \boxed{y = -z}$$

, plug the values in $\eta = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\therefore \eta = \begin{bmatrix} 0 \\ -\lambda \\ \lambda \end{bmatrix} ; \lambda \neq 0$$

So; let $\lambda = 1$

$$\therefore \Rightarrow \eta = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \text{Eigen vector of } \lambda = 1 \text{ of Matrix } A. \quad \text{--- (A)}$$

Also;

Eigen vectors for $\lambda = 2$:

$$\text{Solve: } (A - 2I) \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow (A - 2I) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow (A - 2I) = \begin{bmatrix} 2-2 & 1-0 & 1-0 \\ 1-0 & 3-2 & 2-0 \\ -1-0 & 1-0 & 2-2 \end{bmatrix}$$

$$\Rightarrow (A - 2I) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \rightarrow \text{(b)}$$

Now,

$$\text{to solve : } (A - 2I)\eta = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the Matrix $A - 2I$,

i.e.:

$$A - 2I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

Reduce by Row-Echelon form;

$$\therefore A - 2I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

cancel leading coefficient in R_3
by performing: $R_3 \leftarrow R_3 + 1R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

#1
(swap)

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

cancel leading co-efficients in R_3

by performing: $R_3 \leftarrow R_3 - \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

→ Row echelon form
of $(A-2I)$.

Now:

Reduce the above matrix to
Reduced Row Echelon form

i.e.:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

cancel leading co-efficient
in Row R_1 by performing

$$R_1 \leftarrow R_1 - 1R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Reduced Row Echelon form.}$$

Now, the system associated with the Eigen value $\lambda = 2$.

$$(A - 2I) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (A - 2I) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow (A - 2I)(\eta) = \begin{bmatrix} x + 0 + z \\ 0 + y + z \\ 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} x + z \\ y + z \\ 0 \end{bmatrix}$$

This Reduces to following system of equations;

$$\begin{cases} x + z = 0 \\ y + z = 0 \end{cases} \Rightarrow \begin{cases} x = -z \\ y = -z \end{cases}$$

Plug it into $\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\Rightarrow \eta = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix}, z \neq 0$$

$$\therefore \text{let } z = 1$$

$$\Rightarrow \eta = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \longrightarrow \textcircled{B} : \text{Eigenvector of } \lambda = 2 \text{ of Matrix } A.$$

Similarly:

Eigenvectors for $\lambda = 4$:-

$$\text{Solve: } (A - \lambda I) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2-4 & 1-0 & 1-0 \\ 1-0 & 3-4 & 2-0 \\ -1-0 & 1-0 & 2-4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix} \longrightarrow \textcircled{C}$$

Now, to solve ~~equation~~ $(A - 4I)\eta = 0$

we need to reduce matrix

$$(A - 4I).$$

i.e:

$$A - 4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Reduce matrix to Row Echelon form;

$$\text{i.e.} \begin{bmatrix} a & \text{---} & b \\ 0 & \diagdown & | \\ 0 & 0 & c \end{bmatrix}$$

$$\therefore A - 4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

cancel leading co-efficient in R_2
by performing $R_2 \leftarrow R_2 + \frac{1}{2}R_1$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{5}{2} \\ -1 & 1 & -2 \end{bmatrix}$$

cancel leading co-efficient in R_3
by performing $R_3 \leftarrow R_3 - \frac{1}{2}R_1$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}$$

cancel leading co-efficient in row R_3 by performing $R_3 \leftarrow R_3 + 1R_2$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Row Echelon form of } A - 4I$$

Now, Reduce matrix to Reduced

Row Echelon form; $\begin{bmatrix} 1 & \text{---} & b \\ 0 & \diagdown & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftarrow -2 \cdot R_2$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

cancel leading co-efficient in row R_1
by performing $R_1 \leftarrow R_1 - 1R_2$

$$\sim \begin{bmatrix} -2 & 0 & 6 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftarrow -\frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Reduced Row Echelon form.}$$

Now, The system associated with the Eigen values $\lambda = 4$.

$$(A - 4I)\eta = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \text{As } (A - 4I)\eta = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + 0 - 3z \\ 0 + 1y - 5z \\ 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This reduces to the following system of equations;

$$\begin{cases} x - 3z = 0 & \Rightarrow x = 3z \\ y - 5z = 0 & \Rightarrow y = 5z \end{cases}$$

Plug these values into

$$\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \eta = \begin{bmatrix} 3z \\ 5z \\ z \end{bmatrix} \quad z \neq 0$$

$$\therefore \text{let } z = 1$$

$$\Rightarrow \eta = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}.$$

Hence; The Eigenvectors for $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$

$$\text{are; } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

$$\text{or} \\ \text{Eigenvectors of: } \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

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Assignment : Linear Algebra