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7966 Sec-B

DIFFERENTIAL EQUATION

CIVIL 4th SEM.

FINAL EXAM

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MA'AM SHUMAILA

Question # 02

Answer: (i)

$w = \sin(x+ct) + \cos(2x+2ct)$

$$\frac{\partial w}{\partial t} = \cos(x+ct) + c - \sin(2x+2ct) + 2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) + c^2 - \cos(2x+2ct) + 4c^2 \quad \text{—} (1)$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct) + 2$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$= \left[-\sin(x+ct) - 4\cos(x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial t^2} = +c^2\left[-\sin(x+ct) - 4(\cos(2x+2ct)\right]$$

$c^2 . \partial^2 w / \partial x^2$ Ans.

Hence, $\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}$

Wave Equation:-

The wave equation is an Important second-order linear partial differential equation for the description of wave as they occur in classical physics such as mechanical wave.

Question # 02

Answer:-

Given function is

$$f(x) = \begin{cases} x \ ; & -\pi < x \le 0 \\ 2x \ ; & 0 \le x \le \pi \end{cases}$$

find the fourier co-efficient a_0, a_n & b_n

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$a_0 = \frac{-\pi}{2} + \pi = \frac{\pi}{2} \quad \text{---(1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x(\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi} (2x\cos nx)\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

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$$A_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2} + \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

So

$$a_n = \begin{cases} \frac{-2}{\pi n^2} & ; \text{if } n \text{ in odd} \\ 0 & ; \text{if } n \text{ is even} \end{cases} \rightarrow 2$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi\cos\pi}{n}\right] + \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] \Rightarrow \frac{-3\cos^n\pi}{n}$$

$$= +\frac{3(-1)^{n+1}}{n} \longleftarrow 3$$

fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Answer:- (ii)

$w = \tan(2x+ct)$

Solution:-

As we know that

$w = \tan(2x+ct)$

Diff w.r.t (t)

$\Rightarrow \frac{\delta w}{\delta t} = \frac{\delta}{\delta t} \tan(2x+ct) \cdot c$

$\Rightarrow \frac{\delta w}{\delta t} = \sec^2(2x+ct) \cdot c$

$\Rightarrow \frac{\delta w}{\delta t} = c \sec^2(2x+ct)$

Again diff

$= \frac{\delta^2 w}{\delta t^2} = c \frac{\delta}{\delta t} \sec^2(2x+ct)$

$\Rightarrow \frac{\delta^2 w}{\delta t^2} = c \cdot 2\sec(2x+ct) \cdot \sec(2x+ct) \cdot \tan(2x+ct)$

$\Rightarrow \frac{\delta^2 w}{\delta t^2} = 2c^2 \sec^2(2x+ct) - \tan(2x+ct)$

Now $w = \tan(2x+ct)$

Take diff w.r.t x

$= \frac{\delta w}{\delta x} = \frac{\delta}{\delta x} \tan 2x + ct \cdot$

$$\frac{\delta w}{\delta x} = \sec^2(2x+ct)\cdot 2$$

$$\frac{\delta w}{\delta x} = 2\sec^2(2x+ct)$$

Diff

$$\Rightarrow \frac{\delta^2 w}{\delta x^2} = 2\frac{\delta}{\delta x}\sec^2(2x+ct)$$

$$\Rightarrow \frac{\delta^2 w}{\delta x^2} = 6\sec^2(2x+ct)\tan(2x+ct)$$

As we know that.

$$\frac{\delta^2 w}{\delta t^2} = \frac{c^2\,\delta^2 w}{\delta x^2}$$

putting values in wave equation.

$$1 \neq 3$$

so we have

$$L.H.S \neq R.H.S$$

Hence undefined.

Question # 03

$y'' - 4y' + 13y = 8\sin 3x$. —① $\quad y(0) = 1$, $y'(0) = 2$

Solution:- Associated homogenous eq of ① is

$y'' - 4y' + 13y = 0$ —②

Change ② into Auxiliary equation.

put $y = m$ in ②

$m^2 - 4m + 13 = 0$

Use quadratic formula.

$a = 1, b = -4, c = 13$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36i}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

$m_1 = 2 + 3i$

$m_2 = 2 - 3i$

$y_c = e^{2x}(C_1\cos 3x + C_2 \sin 3x) \rightarrow$ A

Let. $y_p = A\cos 3x + B\sin 3x$ — (*)

Diff w.r.t x

$y_p' = -3A\sin 3x + 3B\cos 3x$

Again diff

$y_p'' = -9A\cos 3x - 9B\sin 3x$ put in (1)

$\Rightarrow (-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x + 3B\cos 3x)$
$+ 13(A\cos 3x + B\sin 3x) = 8\sin 3x)$

$\Rightarrow (-9A - 12B + 13A)\cos 3x + (-9B + 12A + 13B)\sin 3x$
$= 8\sin 3x$

$\Rightarrow (4A - 12B)\cos 3x + (4B + 12A)\sin 3x = 8\sin 3x$

comparing co-efficients

$\sin 3x \Rightarrow \quad 4B + 12A = 8$ — (a)

$\cos 3x \Rightarrow \quad 4A - 12B = 0 \Rightarrow 4A = 12B$

$\boxed{A = 3B}$ — (b)

put (b) in (a)

$4B + 12(3B) = 8$

$4B + 36B = 8$

$40B = 8$

$B = \frac{1}{5}$ — (c)

put (c) in (b)

$\Rightarrow A = 3/5$ — (d)

put c and d in (*)

$$y_p = \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \rightarrow (B)$$

General solution is:

$$y = y_c + y_p$$

$$y = e^{2x}(c_1\cos 3x + c_2 \sin 3x) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \rightarrow (C)$$

Now we need to find the values of c_1 and c_2 for this

put $x=0$, $y=1$ in (C)

$$1 = e^{x(2)}(c_1\cos 3(0) + c_2 \sin 3(0)) + \frac{3}{5}\cos 3(0) + \frac{1}{5}\sin 3(0)$$

$$1 = (c_1(1) + c_2(0)) + 3/5(1) + \frac{1}{5}(0)$$

$$1 = c_1 + 3/5$$

$$c_1 = 1 - 3/5$$

$$c_1 = 2/5 - (**)$$

Diff (C) w.r.t x

$$y' = c_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + c_2(2e^{2x}\sin 3x + 3e^{2x}\cos x) - \frac{6}{5} + \frac{3}{5}\cos 3x \; - (D)$$

put $y' = 2$, $x = 0$ in D

$$y' = c_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + c_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) - 6/5\sin 3x + 3/5\cos 3x.$$

put $y' = 2$, $x = 0$

$$2 = c_1(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0))$$
$$+ c_2(2e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0))$$
$$-6/5 \sin 3(0) + \frac{3}{5}\cos 3(0)$$

$$2 = c_1(2) + c_2(3) - 0 + 3/5$$

$$2 = 2c_1 + 3c_2 + 3/5$$

put $c_1 = 2/5$

$$2 = 4/5 + 3c_2 + 3/5$$

$$2 = 7/5 + 3c_2$$

$$3c_2 = 2 - 7/5$$

$$3c_2 = 3/5$$

$$c_2 = 3/15 \quad \text{———} \quad ***$$

put ** and *** in ©

$$y = e^{2x}\left(2/5 \cos 3x + 3/15 \sin 3x\right) + \frac{3}{5}\cos 3x$$
$$+ \frac{1}{5}\sin 3x$$

$$y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3x$$
$$+ \frac{1}{5}\sin 3x$$

Required general solution

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Question # 04

$(D^2 - DD')z = \cos x \cos 2y$

Solution:-

$(D^2 - DD')z = \cos x \cos 2y$

In CF is given by:

$CF = \phi_1(y) + \phi_2(y+x)$

which its PI is given by

$$PI = \frac{1}{(D^2 - DD')} \cdot \frac{1}{2}\left[\cos(x+2y) + \cos(x-2y)\right]$$

$$= \frac{1}{2}\left(\frac{1}{(-1+2)} \cos(x+2y) + \frac{1}{(-1-2)} \cos(x-2y)\right)$$

$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Hence the complete solution of given PDE is given by:

$$u = \phi_1(y) + \phi_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$