Differential Equations

Name :- Afrasiyab

ID No :- 7899

Section :- A

Assigment No :- 02

Teacher :- Shumail Mazhar

Date :- 15/6/2020

Question # 01

The cauchy Euler equation

(1) $x^3 y''' + 2x^2 y' + 2y - 10x + \frac{10}{x}$

Solution:-

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$$

$$x^3 D^3 y + 2x^2 D^2 + 2y = 10x + 10x^{-1}$$

$$(x^3 D^3 + 2x^2 D + 2)y = 10x + 10x^{-1} \quad (1)$$

let $x = et \Rightarrow t = \ln x$

$$xD = D$$

$$x^2 D^2 = D(D-1) = D^2 - D$$

$$x^3 D^3 = D(D-1)(D-2)$$

Subtituting into eq (1)

$$(D - 3D^2 + 2D + 2(D^2 - D) + 2)y = 10^x + 10x^{-1}$$

$$(D^3 - D^2 + 2)y = 10x + 10x^{-1}$$

$$(m^3 - m^2 + 2)y = 10e^t + \frac{10}{et}$$

using synthtic division

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 0 & 2 \\ & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$D^2 - 2D + 2 = 0$

Now using cluadratic formula

$a = 1,\ b = -2,\ c = 2$

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a}$$

$$D = \frac{2 \pm \sqrt{4-8}}{2}$$

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-1} + \sqrt{4}}{2}$$

$$D = \frac{2 + 2c}{2}$$

$$D = \frac{\cancel{2}(1 \pm 2)}{\cancel{2}}$$

$$D = 1 \pm i$$

since roots ass Complex

$y_c = e^{-x}(c_1 \cos t + c_2 \sin t)$

Now Posticular integration

$$yp = \frac{1}{D^3 - D^2 + 2} 10e^t + \frac{1}{D^3 - D^2 + 2} 10/e^t$$

$$= \frac{10e^t}{(1)^3 - (1)^2 + 2} + \frac{10e^{-t}}{(1)^3 - (1)^2 + 2}$$

$$= \frac{\cancel{10}^{5} e^t}{\cancel{2}} + \frac{\cancel{10}^{5} e^{-t}}{\cancel{2}}$$

$$= 5e^t + 5e^t$$

$$yp = 5e^t + 5e^t$$

General Solution

$$y = yc + yp$$

$$y = e^{-x}(c_1 \cos t + c_2 \sin t) + 5e^t + 5e^t$$

Put $e^t = x$ and $t = \ln x$

$$\boxed{y = e^{-x}(c_1 \ln x + c_2 \sin \ln x) + 5e^x + 5e^{-x}}$$

Ans.

Question : 02

2) $x^3 \frac{d^3y}{dx} + 4x^2 \frac{d^2y}{dx} - 5x\frac{dy}{dx} - 15y = x^4$

Solution :-

Let $\frac{d}{dx} = D$

$x^3D^3y + 4x^2D^2y - 5xDy - 15y = x^4$

$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$

Let $\cancel{x}\ x = e^t \Rightarrow t = \ln x$

$xD = D$

$x^2D^2 = D(D-1) = D^2 - D$

$x^3D^3 = D(D-1) = (D^2 - 2D) = D^3 - 3D^2 + 2D$

Now subtituting

$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$

$(D^3 - \cancel{3D}\ 3D^2 + 2D + 4(D^2 - D) - 5(D) - 15)y$

e^{4t}

$(D^3 + D^2 - 7D - 5)y = e^{4t}$

Synthetic division :-

5	1	+1	-7	-15
		3	12	15
	1	4	5	0

$D^2 + 4D + 5 = 0$

~~chat~~ cluactratic formula

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \cancel{+} -4 \pm 2i$$

$$D = \frac{\cancel{D}(-2 \pm 2)}{2}$$

$$y_c = e^{5n}(c_1 \cos t + c_2 \sin t)$$

for $yp = ?$

$$yp = \frac{1}{D^3 + D^2 - 7D - 15} e^{4t}$$

$$= \frac{1}{(4)^3 + (4^2) - 7(4) - 15} e^{4t}$$

$$= \frac{1}{64 + 16 - 28 - 15} e^{4t}$$

$$= \frac{1}{80 - 43} e^{4t}$$

(6)

$$y_P = \frac{1}{37} e^{4t}$$

Hence

$$y = y_C + y_P$$

$$y = (c_1 \cos t + c_2 \sin t) + \frac{1}{37} e^{4t}$$

again put $t = \ln x$ and $x = \ln x$

$$\boxed{y = e^{34} (c_1 \cos t \ln x + c_2 \sin \log x) + \frac{1}{37} e^{4t}}$$

$\therefore$ Ans

$\longleftrightarrow$

Question:03

$x^2y'' + 2xy' - 6y = 10x^2$

Solution:-

$y(1) = 1$ and $y'(1) = -6$

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2$$

$$\Rightarrow \left(x^2 d^2/dx^2 + 2x\, d/dx - 6\right)y = 10x^2$$

Put $xD = D \Rightarrow x^2D^2 = D(D-1) = D^2 - D$

$x = e^t$ and $\log x = t$

$(D^2 - D + 2D - 6)y = 10e^{2t}$

$(D^2 + D - 6)y = 10e^{2t}$

The characteristic equation

$D^2 + D - 6 = 0$

$D^2 + 3D - 2D - 6 = 0$

$\Rightarrow D(D+3) - 2(D+3) = 0$

$\Rightarrow (D+3)(D-2) = 0$

Scanned with CamScanner

8

$D = 2, D = -3$

Since roots as real and distinct

for yc =?

$y_c = c_1 e^{-3t} + c_2 e^{2t}$

for yp =?

$$y_p = \frac{1}{D^2 - D - 6} 10^{2t}$$

$$= \frac{10}{D^2 - D - 6} e^{2t}$$

$$= 10 \frac{1}{0} e^{2t} \text{ fails}$$

Now

$$10 \frac{1}{d/dz (D^2 + D - 6)} e^{2t}$$

$$\Rightarrow 10 \frac{t}{2D+1} e^{2t}$$

$$\Rightarrow 10 \frac{1 \cdot t}{4+1} e^{2t}$$

$y_p = 2te^{2t}$

Scanned with CamScanner

9

General solution

$y = y_c + y_p$

$= c_1 e^{-3t} + c_2 e^{2t} + 2te^{2t}$

$y = c_1 x^{-3} + c_2 x^2 + 2(\log x)x^2$ - (B)

Put $y(1) = 1$ i.e $x = 1, y = 1$ in (B)

$1 = c_1 (1)^{-3} + c_2 (1)^2 + 2\log(1)$

$1 = c_1 + c_2 \rightarrow$ (C)

Now differentiate eq (B) w.r.t x.

$y' = -3c_1 x^{-4} + 2c_2 x + \frac{2}{x}(x^2) + 4x\log x$

Now put $y'(1) = -6$ i.e $y' = -6$ and $x = -6$

$-6 = -3c_1 + 2c_2 + 2 + 0$

$\Rightarrow -6 = -3c_1 + 2c_2 + 2$

$-8 = -3c_1 + 2c_2$ - (D)

Xing eq (C) with (2) and -ing from (D)

$2c_1 + 2c_2 = 2$

$\mp 3c_1 \pm 2c_2 = \mp 8$

$5c_1 = 10$

$c_1 = \frac{10}{5} = 2$ $\quad \boxed{c_1 = 2}$

Scanned with CamScanner

$-8 = -3(2) + 2C_2$

$-8 = -6 + 2C_2$

$2C_2 = -8 + 6$

$2C_2 = -2$

$C_2 = \frac{-2}{2}$ 1

Now put the value of c_1 and C_2 in eq (B)

$y = 2x^{-3} - x^2 + 2 \ln x \; x(x^2)$

$\boxed{y = \frac{2}{x^3} - x^2 + 2x^2 \log x}$ Ans

Question = 04

$x^2y'' + 7xy' + 5y = x^5$

$y(0) = 2$ and $y'(1) = 2$

Solution :-

$$x^2\frac{dy^2}{dx^2} + 7x\frac{dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2\frac{d^2}{dx^2} + 7x\frac{d}{dx} + 5\right)y = x5 \quad -(A)$$

Put $xD = D \Rightarrow x^2D^2 = D(D-1) = D^2 - D$

$x = e^t \Rightarrow \log x = t$ in eq (A)

$$\Rightarrow (D^2 - D + 7D + 5)y = e^{5t}$$

$$\Rightarrow (D^2 + 6D + 5)y = e^{5t}$$

By quadratic formula.

$$D = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$= \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{-6 \pm \sqrt{4^2}}{2}$$

$$= \frac{2(-3 \pm 2)}{2}$$

$D = -3 \pm 2$ since roots are real and distinct

$y_c = C_1 e^{-5t} + C_2 e^{-t}$

for $y_P = ?$

$$y_P = \frac{1}{D^2 + 6D + 5} e^{5t}$$

$$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$$

$$= \frac{1}{60} e^{5t}$$

Now General solution is

$y = y_c + y_P$

$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} \times 5 \rightarrow$ B

$x = 0$ put in This equation ~~No in eq~~

No in eq (B) $e^{0} = 1$

Put $y(0) = 2$, $-ey = 2$ and $x = 2$

$$2 = C_1 (2)^{-5} + C_2 (2)^{1} + \frac{1}{60}(2)^5$$

$$2 = -32 C_1 - 2 C_2 + \frac{1}{\cancel{60}_{15}} (\cancel{32}^{8})$$

$$2 = -32 C_1 - 2 C_2 + \frac{8}{15}$$

$$2 - \frac{8}{15} = -32 C_1 - 2 C_2$$

$$\frac{22}{15} = -32 C_1 - 2 C_2 \longrightarrow (C)$$

Now diffesantiale eq (B) w.r.t (x)

$$y' = -5c_1 x^{-6} - C_2 x^{-2} + \frac{1}{12} x^4 \longrightarrow$$

Put $y'(1) = 2$, $-ey' = 2$ and $x = 2$ in above equation

$$2 = -5 C_1 (2)^{-6} - C_2 (2)^{-2} + \frac{1}{12}(2)^4$$

$$2 = -5c_1(-64) - c_2(4) + \frac{1}{12}(16)$$

$$2 = 320 C_1 + 4 C_2 + 4/3$$

$\Rightarrow 2 - \frac{4}{3} = 620 C_1 + 4 C_2$

$\Rightarrow 2/3 = 320 C_1 + 4 C_2 \rightarrow$ D

Xing eq C with 2 and then -ing eq (c) from D

$\frac{-44}{15} = 64 C_1 + 4 C_2)$

$\frac{-44}{15} = 64 C_1 + 4 C_1$

$+ 2/3 = \pm 320 C_1 \pm 4 C_2$

$\frac{34}{15} = -256 C_1$

$C_1 = \frac{34}{15} \times 256$

$\boxed{C_1 = 580}$

Put The value of C_1 in eq (c)

$\frac{22}{15} = -32(580) - 2 C_2$

$\Rightarrow \frac{22}{15} = -18560 - 2 C_2$

$\Rightarrow \frac{22}{15} + 18560 = -2 C_2$

$\Rightarrow \frac{18561}{-2} = C_2$

$\boxed{-9280 = C_2}$

Scanned with CamScanner

Now put the value of C_1 and C_2 in eq (B)

$$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$$

$$\boxed{y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5}$$

Ans

Question . 05

$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$

Solution:-

$$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$\Rightarrow \left((x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right)y = x^2$$

$$\Rightarrow [(x+1)^2 D^2 - 3(x+1)D + 4]y = x^2 \rightarrow (A)$$

Put $(x+1)D = D \Rightarrow (x+1)^2 D^2 = D(D-1) = D^2 - D$

$x = e^t$ in eq (A)

$$\Rightarrow [D^2 - D - 3D + 4)y = e^{2t}$$

$$\Rightarrow [D^2 - 4D + 4)y = e^{2t}$$

$$\Rightarrow (D^2 - 4D + 4)2 = e^{2t}$$

for y_c we find the roots

$$D^2 - 4D + 4 = 0$$

$$D^2 - 2D - 2D + 4 = 0$$

$$D(D-2) - 2(D-2) = 0$$

$D-2=0, \; D=2$

$D-2=0, D=2$

So the roots asered and reapet the

Genesal solutionase

$y=(c_1+c_2x)^{2x}$

$y=(c_1+c_3x)^{2x}$

for up=?

$yp \; \frac{1}{D^2-4D+4}$ $\quad (2)^2-4(2)+4$

$=>0$

$yp = \frac{2}{2D-4} e^{2t}$

if we put 2⁽¹⁾

$2D-4 => 2(2)-4=0$

we take again desivatic

$yp = \frac{2}{2} e^{2t}$

$\boxed{y=(c_1+c_2x)^{2y}+e^{2t} \rightarrow \text{Geneal}}$

salution.

Ans