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Assignment :→ Differential Equation

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Cauchy Euler Method

Q NO :→ (01)

$$x^3y''' + 2x^2y' + 2y = 10x + \frac{10}{x}$$

Solution

$$x^3y''' + 2x^2y' + 2y = 10x + \frac{10}{x} \rightarrow (1)$$

put $x = e^t$ then

$$\frac{dx}{dt} = e^t \Rightarrow \frac{dt}{dx} = e^{-t}$$

Now

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot e^{-t}$$

OR

$$y' = \frac{dy}{dx} = e^{-t} Dy \qquad \because \frac{d}{dt} \rightarrow D$$

P.T.O

Similarly

$$y'' = e^{-2t}\,[D(D-1)]\,y$$

$$y''' = e^{-3t}\,[D(D-1)(D-2)]\,y$$

Using these values in ①

$$e^{3t}\cdot e^{-3t}\,[D(D-1)(D-2)]\,y + 2e^{2t}\cdot e^{-2t}\,[D(D-1)]\,y + 2y = 10e^{t} + 10e^{-t}$$

$$\Rightarrow (D^3 - 3D^2 + \cancel{2D})\,y + (2D^2 - \cancel{2D})\,y + 2y = 10e^{t} + 10e^{-t}$$

$$\Rightarrow D^3y - D^2y + 2y = 10e^{t} + 10e^{-t}$$

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 10e^{t} + 10e^{-t} \rightarrow ②$$

The assosiated homogenious equation of ②

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 0$$

Say, $\frac{d}{dt^2} = K^2$, $\frac{d^3}{dy^3} = K^3$

$$\Rightarrow (K^3y - K^2y + 2y) = 0$$

$$\Rightarrow (K^3 - K^2 + 2)\,y = 0$$

For non-trivial Sol., $y \neq 0 \Rightarrow$

$$K^3 - K^2 + 2 = 0$$

P.T.O

⇒ Roots are
$$K = -1,\ 1 \pm i$$

⇒ $y_c(t) = Ae^{-t} + (B\cos t + C\sin t)e^{t}$

when is Complementry Sol.

Q NO:→ (03)

$$x^2y'' + 2xy' - 6y = 10x^2 : y(1) = 1$$
$$y'(1) = -6$$

Solution

$$x^2y' + 2xy' - 6y = 10x^2 \rightarrow ①$$

let $\qquad y(1) = 1$

$x = e^t$ i.e $t = \log x \qquad y'(1) = -6$

Now

$$xy' = \Delta y \Rightarrow x^2y' = \Delta(\Delta - 1)y$$

where $\Delta = \dfrac{d}{dt}$

Then equation ① ⇒ $[\Delta(\Delta-1) + 2\Delta - 6]y = 10e^{2t}$

$$[\Delta^2 - \Delta + 2\Delta - 6]y = 10e^{2t}$$

P.T.O

$\Rightarrow [\Delta^2 + \Delta - 6]y = 10e^{2t}$

Char form Equation $\Delta^2 + \Delta - 6 = 0$

$\Delta + 3\Delta - 2\Delta - 6 = 0$

$\Delta = -3, \ \Delta = 2$

Complementry function

$C.F = C_1 e^{-3t} + C_2 e^{2t}$

Also P. Integral

$P.I = \frac{1}{\Delta^2 + \Delta - 6} 10e^{2t}$

$= 10 \frac{1}{(2)^2 + 2 - 6} e^{2t}$ replace Δ by 2

Case of failure

$P.I = 10.t. \frac{1}{2\Delta + 1} e^{2t} = 10t \cdot \frac{1}{2(2)+1} e^{2t}$

$= 10t \ \frac{1}{5} e^{2t} = 2t e^{2t}$

Hence general Sol. $y = C.F + P.I$

$y = C_1 e^{-3t} + C_2 e^{2t} + 2t e^{2t}$

$C_1 x^{-3} + C_2 x^2 + 2(\log x) x^2$

P.T.O

Apply Initial condition

$y(1) = 1$ we get

$1 = c_1 + c_2 + 0 \to$ (A)

& $y'(1) = -6$

$y' = -3c_1 x^{-4} + 2c_2 x + 2x + 4x \log x$

$-6 = 3c_1 + 2c_2 + 2 + 0$

$-3c_1 + 2c_2 = -8 \to$ (B)

eq (A) × 3 and add with (B)

$3 = \cancel{3c_1} + 3c_2$

$-8 = \cancel{-3c_1} + 2c_2$

$5c_2 = -5$

$\boxed{c_2 = -1}$

eq (A) → $1 = c_1 - 1$

$\boxed{c_1 = 2}$

thus ⇒ $\boxed{y = 3x^{-3} - x^2 + 2x^2 \log x}$

P.T.O

Q NO:→ (04)

$$x^2y'' + 7xy' + 5y = x^5;$$

$$y(0) = 2 \ \& \ y'(1) = 2$$

Solution

let

$$x = e^t \Rightarrow t = \log x, \ \Delta = \frac{d}{dt}$$

Now $xy' = \Delta y \Rightarrow x^2y'' = \Delta(\Delta - 1)y$

then

$$(\Delta(\Delta-1) + 7\Delta + 5)y = e^{5t}$$

$$(\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}$$

$$(\Delta^2 + 6\Delta + 5)y = e^{5t}$$

Char eq is $\Delta^2 + 6\Delta + 5 = 0$

$$\Delta^2 + 5\Delta + \Delta + 5 = 0$$

$$\Delta = -5, \ -1$$

Complementare eq is

$$C.F = C_1 e^{-5t} + C_2 e^{-t}$$

P.T.O

→ P. Integral

$$P.I. = \frac{1}{\Delta^2+6\Delta+5} e^{5t}$$

$$= \frac{1}{5^2+6(5)+5}$$ Replacing Δ by 5

$$= \frac{1}{60} e^{5t}$$

Thus

$$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t}$$

$$y = C_1 x^{-5} + C_2 x^{-1} + \frac{1}{60} x^5$$

$$y' = -5C_1 x^{-6} - C_2 x^{-2} + \frac{1}{12} x^4$$

$y(0) = 2$ $\quad x = 0, \; y = 2$

$$2 = C_1 + C_2 + \frac{1}{60}$$

$$C_1 + C_2 = 119/60 \rightarrow (A)$$

$y'(1) = 2$ $\quad x = 1, \; y' = 2$

$$2 = -5C_1 - C_2 + 1/12$$

$$-5C_1 - C_2 = 23/12 \rightarrow (B)$$

$$A + B \quad -4C_1 = \frac{234}{60} \Rightarrow 4 = \frac{-117}{120}$$

Now

$$y = \frac{-117}{120} x^{-5} + C_2 x^{-1} + \frac{1}{60} x^5$$

$$C_1 = \frac{-117}{120} \text{ put in } (A) \quad \frac{-117}{120} + C_2 = \frac{119}{60}$$

$$C_2 = \frac{119}{60} + \frac{117}{120}$$

$$\boxed{= \frac{238 + 117}{120} = \frac{355}{120}}$$

Q NO: (05)

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$$

Solution

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2 \longrightarrow (1)$$

let

$$x+1 = e^t \Rightarrow x = e^t - 1$$

Diff $\log(x+1) = t$

Also $(x+1)y' = \Delta y$, $\left\{ \frac{d}{dt} = \Delta \text{ and } D = \frac{d}{dx} \right\}$

P.T.O

Then eq ① ⇒ $(\Delta(\Delta-1)-3\Delta+4)y=(e^t-1)^2$

$$(\Delta^2-4\Delta+4)y=e^{2t}-2e^t+1$$

Char eq is $\Delta^2-4\Delta+4=0$

$$(\Delta-2)^2=0$$

$$\Delta=2,2$$

This Complementry function is

$$C.F=(C_1+C_2t)e^{2t}$$

Also particular Integral is

$$P.I=\frac{1}{(\Delta-2)^2}(e^{2t}-2e^t+1)$$

$$=\frac{1}{(\Delta-2)^2}e^{2t}-2\frac{1}{(\Delta-2)^2}e^t+\frac{1}{(\Delta-2)^2}\rightarrow ②$$

Now

$$\frac{1}{(\Delta-2)^2}e^{2t}=\frac{1}{(2-2)^2}e^{2t}=\frac{1}{0}e^{2t}$$

Case of failure

$$\frac{1}{(\Delta-2)^2}e^{2t}=t\frac{1}{2(1-2)^2}e^t=\frac{t^2e^t}{2}$$

and $2\frac{1}{(\Delta-2)^2}e^t=2\frac{1}{(1-2)^2}e^t=2e^t$

P.I.O

and $\frac{1}{(\Delta-2)^2}(1) = \frac{1}{(\Delta-2)^2} e^{0t} = \frac{1}{4}$

eq ② $\Rightarrow$ $P.I = \frac{1}{2} t^2 e^{2t} - 2e^t + \frac{1}{4}$

Hence Complete Solution is

$y = C.F + P.I$

$y = (C_1 + C_2 t) e^{2t} + \frac{1}{2} t^2 e^{2t} - 2e^t + \frac{1}{4}$

Repect Value of e^t

$y = (C_1 + C_2 \log(x+1)(x+1)^2 + \frac{1}{2} [(\log(x+1))^2$

$(x+1)^2] - 2(x+1) + \frac{1}{4}$

OR

$y = C_1 + C_2 \log(x+1)(x+1)^2 +$

$\frac{1}{2} (\log(x+1)^2 (x+1)^2] - 2x - 7/4$

which is the Required.

P.T.O

Q NO:→ (02)

$$x^3y''' + 4x^2y'' - 5xy' - 15y = x^4$$

Solution

$$x^3y''' + 4x^2y'' - 5xy' - 15y = x^4 \rightarrow (1)$$

let

$$x^2y = e^t \rightarrow x = e^t - 1$$

Diff $\log(xy) = t$

Also

$$(xy) = \Delta y, \quad \left\{ \frac{d}{dt} = \Delta \text{ and } D = \frac{d}{dx} \right\}$$

then eq (1)

$$(\Delta(\Delta-1) - 15\Delta + 5)y = (e^t - 1)^2$$

$$(\Delta^2 - 15\Delta + 5)y = e^{2t} - 2e^t + 1$$

Char eq is $(\Delta^2 - 15\Delta + 5)y = 0$

$$(\Delta - 5)^2 = 0$$

$$\Delta = 5, 5$$

P.T.O

this Complementry function is

$$C.F = (C_1 + C_2 t)\, e^{2t}$$

Also particular Integral is

$$P.I = \frac{1}{(\Delta - 5)^2}\left(e^{2t} - 2e^{t} + 1\right)$$

$$= \frac{1}{(\Delta - 2)^2} e^{2t} - 5\frac{1}{(\Delta - 5)^2} e^{t} + \frac{1}{(\Delta - 5)^2} \rightarrow (2)$$

Now $\frac{1}{(\Delta - 5)^2} e^{2t} = \frac{1}{(5-5)^2} e^{2t} = \frac{1}{0} e^{2t}$

Case of failure

$$\frac{1}{(\Delta - 5)^2} e^{2t} = t \frac{1}{5(1-5)^2} e^{t} = \frac{t^2 e^{t}}{5}$$

and $5\frac{1}{(\Delta - 5)^2} e^{t} = 5\frac{1}{(1-5)^2} e^{t} = 5e^{t}$

and $\frac{1}{(\Delta - 5)^2}(1) = \frac{1}{(\Delta - 5)^2} e^{0t} = \frac{1}{15}$

eq (2) $\Rightarrow P.I = \frac{1}{5} t^2 e^{5t} - 5e^{t} + \frac{1}{15}$

Hence Complete Solution is

P.T.O

$$\Rightarrow \quad y = C.F + P.I$$

$$y = (C_1 + C_2 t)e^{2t} + \frac{1}{5}t^2 e^{2t} - 5e^t + \frac{1}{15}$$

$$y = C.F + P.I$$

$$y = C_1 + C_2 t) e^{2t} + \frac{1}{5} t^2 e^{2t} - 5e^t + \frac{1}{15}$$

Repect Value of e^t

$$y = (C_1 + C_2 \log(x+1)(x+1)^2 + \frac{1}{2}$$

$$\left[(\log(x+1)^2 (x+1)^2\right] - 5(x+1)$$

$$+ \frac{1}{15}$$

~~(x+1)~~

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