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Final Exam

Differential Equations:

Q1: (a)

2nd Order Linear homogeneous/non-homogeneous differential equation.

A differential equation of any order is homogeneous, if once all the terms involving the unknown function are collected together on one side of the equation, the other side is identically zero.

i.e

$y'' - 2y' + y = 0$ is a 2nd order homogeneous DE.

A differential equation of any order is non-homogeneous, if once all the terms involving the unknown functions are collected together on one side, the other side is not identically zero.

i.e

$$y'' - 2y' + y = 4$$

Q 1 : b : Solve the following 2nd order linear homogeneous/ non-homogeneous differential equation:

i, $4y'' - 6y' + 7y = 0$.

Lets find the root of the characteristics equation:

$$4\lambda^2 - 6\lambda + 7 = 0$$

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(4)(7)}}{2(4)}$$

$$\lambda = \frac{6 \pm \sqrt{36 - 112}}{8}$$

$$\lambda = \frac{6 \pm \sqrt{-76}}{8}$$

$$\lambda = \frac{6}{8} \pm \frac{2\sqrt{19}\,i}{8}$$

$$\lambda = \frac{3}{4} \pm \frac{\sqrt{19}}{4} i$$

$$\lambda_1 = \frac{3}{4} + \frac{\sqrt{19}}{4} i, \quad \lambda_2 = \frac{3}{4} - \frac{\sqrt{19}}{4} i$$

so it has the complex conjugate roots:

$$\phi_1(x) = e^{\lambda_1 x} \cos^{\lambda_1 (x)}$$

$$\phi_2(x) = e^{\lambda_2 x} \sin^{\lambda_2 (x)}$$

$$y = c_1 e^{\frac{3x}{4}} \cos^{i\frac{\sqrt{19}}{4}(x)} + e^{\frac{3x}{4}} \sin^{i\frac{\sqrt{19}}{4}(x)} c_2 .$$

Q1 : D :

ii) $y'' - 4y' - 12y = 3e^{5x}$.

Sol:

The characteristic equation and its roots:

$$r^2 - 4r - 12 = (r-6)(r+2) = 0.$$

$$r_1 = -2,\ r_2 = 6.$$

The complementary solution is then:

$$y_c(t) = c_1 e^{-2t} + c_2 e^{6t}.$$

Q2/ Solve the following IVP for the 2^{nd} order linear equations:

i: $16y'' - 40y' + 25y = 0$

$y(0) = 3, \quad y'(0) = -\frac{9}{4}$.

Sol:

The characteristic equation and its roots are:

$16r^2 - 40r + 25 = (4r - 5)^2 = 0 \quad r_1 = \frac{5}{4}, r_2 = \frac{5}{4}$

The general solution and its derivative are

$$y(t) = c_1 e^{\frac{5t}{4}} + c_2 t e^{\frac{5t}{4}}$$

$$y'(t) = \frac{5}{4} c_1 e^{\frac{5t}{4}} + c_2 e^{\frac{5t}{4}} + \frac{5}{4} c_2 t e^{\frac{5t}{4}}$$

putting in the initial condition.

$$3 = y(0) = c_1$$

$$-\frac{9}{4} = y'(0) = \frac{5}{4} c_1 + c_2$$

The solution for IVP is then

$$y(t) = 3e^{\frac{5t}{4}} - 6te^{\frac{5t}{4}}$$

Q2 : part : ii

$$y'' + 14y' + 49y = 0 \quad y(-4) = -1, \quad y'(-4) = 5 .$$

Solution:

The characteristic equation and its roots are:

$$r^2 + 14r + 49 = (r+7)^2 = 0 \quad r_1 = -7, r_2 = -7$$

The general solution and its derivative are:

$$y(t) = c_1 e^{-7t} + c_2 t e^{-7t}$$
$$y'(t) = -7c_1 e^{-7t} + c_2 e^{-7t} - 7c_2 t e^{-7t}$$

putting in the intial condition:

$$-1 = y(-4) = c_1 e^{28} - 4c_2 e^{28}$$
$$5 = y'(-4) = -7c_1 e^{28} + c_2 e^{28} + 28c_2 e^{28}$$
$$= -7c_1 e^{28} + 29c_2 e^{28}$$

It gives the following constants by solving:

$$c_1 = -9e^{-28}$$
$$c_2 = -2e^{-28}$$

The solution for IVP is

$$y(t) = -9e^{-28}e^{-7t} - 2te^{-28}e^{-7t}.$$

$$y(t) = -9e^{-7(t+4)} - 2te^{-7(t+4)}.$$

Q2 : Part : iii

$$y'' - 4y' + 9y - 0 \qquad y(0) = 0, \; y'(0) = -8$$

Sol:

The characteristic equation for this DE is :

$$r^2 - 4r + 9 = 0.$$

The roots of the equation are

$$r_1 = 2 + \sqrt{5}\, i$$
$$r_2 = 2 - \sqrt{5}\, i$$

The general solution to the differential equation is then

$$y(t) = c_1 e^{2t} \cos(\sqrt{5}t) + c_2 e^{2t} \sin(\sqrt{5}t)$$

Applying initial condition along with derivatives.

$$y(t) = c_2 e^{2t} \sin(\sqrt{5}t).$$
$$y'(t) = 2c_2 e^{2t} \sin(\sqrt{5}t) + \sqrt{5} c_2 e^{2t} \cos(\sqrt{5}t)$$

$$-8 = y'(0) = \sqrt{5} c_2 \Rightarrow c_2 = -\frac{8}{\sqrt{5}}$$

Solution is then

$$y(t) = \frac{-8}{\sqrt{5}} e^{2t} \sin(\sqrt{5}t).$$

Q2: part: iv:

$$y'' - 8y' + 17y = 0 \qquad y(0) = -4, \quad y'(0) = -1.$$

The characteristic equation and its roots are:

$$r^2 - 8r + 17 = 0.$$
$$r_1 = 4 + i$$
$$r_2 = 4 - i$$

The general solution as well as derivative is:

$$y(t) = c_1 e^{4t} \cos(t) + c_2 e^{4t} \sin(t)$$
$$y'(t) = 4c_1 e^{4t} \cos(t) - c_1 e^{4t} \cos(t) + 4c_2 e^{4t} \sin(t) + c_2 e^{4t} \cos(t)$$

By Applying the intial condition it gives the following,

$$-4 = y(0) = c_1$$
$$-1 = y'(0) = 4c_1 + c_2.$$

The solution is then

$$y(t) = -4e^{4t} \cos(t) + 15 e^{4t} \sin(t)$$

Q3: Define Laplace transform along with example?

Ans:

Laplace Transform:

Laplace transform is integral transform that converts a function of real variable (t) to the function of complex variable (s).

i.e

The Laplace transform of a function $f(t)$ for $t>0$ is defined by the following integral over 0-∞.

$$y\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

general example is

$$f(s) = \int_0^\infty f(t) \cdot e^{-st}\,dt.$$

Q3 : A : part : 1

$$f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$$

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

Q3 : A : part 2

$$g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$$

$$G(s) = 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

Q3 : A : part 3

$$h(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t).$$

$$H(s) = \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}.$$

Q4: Solve the following IVP using Laplace Transforms.

i, $y'' - 10y' + 9y = 5t$, $y(0) = -1$, $y'(0) = 2$.

Solution:

Taking transform of every term

$$L\{y''\} - 10L\{y'\} + 9L(y) = L\{5t\}$$

by formulas, we got

$$s^2Y(s) - sy(0) - y'(0) - 10(sY(s) - y(0)) + 9Y(s) = 5/s^2.$$

putting in the initial conditions

$$(s^2 - 10s + 9)Y(s) + s - 12 = \frac{5}{s^2}$$

Solve for $Y(s)$:

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}.$$

The partial fraction of transform will be:

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}.$$

$$5 + 12s^2 - s^3 = A \cdot s(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

Solving for constants:

$s=0$ $\quad 5 = 9B \quad \Rightarrow B = 5/9$

$s=1$ $\quad 16 = -8D \quad \Rightarrow D = -2$

$s=9$ $\quad 248 = 648C \quad \Rightarrow C = 31/81$

$s=2$ $\quad 45 = -14A + \frac{4345}{81} \quad \Rightarrow A = 50/81.$

Plugging in the constant gives:

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

By taking the inverse transform the solution is then:

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^t.$$

Q4: part ii

$$y'' - 6y' + 15y = 2\sin(3t), \quad y(0) = -1, \quad y'(0) = -4$$

Solution:

Taking the laplace transform of everything and plug in intial conditions:

$$s^2Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 15Y(s) = 2\frac{3}{s^2+9}$$

$$(s^2 - 6s + 15)Y(s) + s - 2 = \frac{6}{s^2+9}$$

$$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)}$$

Lets get the partial fraction decomposition:

$$Y(s) = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}$$

now, setting numerators equal gives:

$$-s^3+2s^2-9s+24=(As+B)(s^2-6s+15)+(Cs+D)(s^2+9)$$

$$=(A+C)s^3+(-6A+B+D)s^2+(15A-6B+9C)s+15B+9D.$$

solving for constants:

$$\left.\begin{array}{ll} s^3: & A+C=-1 \\ s^2: & -6A+B+D=2 \\ s^1: & 15A-6B+9C=-9 \\ s^0: & 15B+9D=24 \end{array}\right\} \quad A=\frac{1}{10} \quad B=\frac{1}{10} \quad C=-\frac{11}{10} \quad D=\frac{5}{2}.$$

plugging in the constant gives:

$$Y(s)=\frac{1}{10}\left(\frac{s+1}{s^2+9}+\frac{-11s+25}{s^2-6s+15}\right)$$

$$=\frac{1}{10}\left(\frac{s}{s^2+9}+\frac{\frac{1}{3}3}{s^2+9}-\frac{11(s-3)}{(s-3)^2+6}-\frac{8\frac{\sqrt{6}}{\sqrt{6}}}{(s-3)^2+6}\right).$$

finally, take the inverse transform and solution will be then;

$$y(t)=\frac{1}{10}\left(\cos(3t)+\frac{1}{3}\sin(3t)-11e^{3t}\cos(\sqrt{6}t)-\frac{8}{\sqrt{6}}e^{3t}\sin(\sqrt{6}t)\right)$$