Name Aamar Ghafoor

ID 7728

Subject

Difficerential Equations.

Question No # 1

Part (1)

$W = \sin(x+ct) + \cos(2x+2ct)$

$$\frac{\partial^2 W}{\partial t^2} = c^2 \frac{\partial^2 W}{\partial x^2}$$

Solution:-

$$\frac{\partial w}{\partial t} = \cos(x+ct)+c - \sin(2x+2ct)+2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct)+c^2 - \cos(2x+2ct)+4c^2$$

A

$$\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct)+2$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$= \left[-\sin(x+ct) - 4\cos(x+2ct)\right]$$

$$\frac{\partial w}{\partial t^2} = +c^2\left[-\sin(x+4) - 4(\cos(2x+2ct)\right]$$

$$c^2 \cdot \frac{\partial^2 w}{\partial x^2}$$

(ii) $w = \tan(2x+ct)$

solution:.

Now $\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left(c\sec^2(2x+ct)\right)$

$= c^2\, 2\sec^2(2x+ct)\tan(2x+ct)$

Now $\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$

$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$

$= 4c^2 \cancel{\sec^2}(2x+ct)\tan(2x+ct) = 4c^2 \cancel{\sec^2}(2x+ct)\tan(2x+ct)$

$0 = 0$ satisfied

Q No #2

Expand the following functio in fourier Series.

$$F(x) = x, \quad -\pi < x \leq 0$$
$$= 2x, \quad 0 \leq x \leq \pi$$

Solution:-

Given function is

$$\begin{cases} x ; & -\pi < x \leq 0 \\ 2x, & 0 \leq x \leq \pi \end{cases}$$

Now we have to find the fourier co-efficent, a_0, a_n & b_n

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$\boxed{a_0 = -\frac{\pi}{2} + \pi = \pi/2} \text{ — eq (1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin x}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin x}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n+2(-1)^n-2}{n^2}\right] = \frac{(-1)^n-1}{\pi n^2}$$

So

$$a_n = \begin{cases} \frac{-2}{\pi n^2}; & \text{if } n \text{ is odd} \\ 0 ; & \text{if } n \text{ is even} \end{cases} \rightarrow (2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin n(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$\boxed{b_n = \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{-3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}} \rightarrow (3)$$

So required the Fourier Series is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Q No # 3

Solve the initial Value Problem

$$y'' - 4y' + 13y = 18\sin 3x$$
$$y(0) = 1 \text{ and } y'(0) = 2$$

Solution:.

Associated Homogenous Eq of ① is

$$y'' - 4y' + 13y = 0 \rightarrow ②$$

Change ② into Auxilary equation:-

put $y = m$ in ②

$$m^2 - 4m + 13 = 0$$

Use Quadatic formula

$$a = 1, \quad b = -4, \quad c = 13$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36}\, i}{2}$$

$$= \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$m_1 = 2 + 3i$$
$$m_2 = 2 - 3i$$

$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$ — (A)

put

$y_p = A \cos 3x + B \sin 3x \rightarrow (x)$

Diff w.r.t "x"

$y''_p = -9A \cos 3x - 9B \sin 3x$ put in eq (1)

$(-9A \cos 3x - 9B \sin 3x) - 4(-3A \sin 3x + 3B \cos^3 x) + 13$
$(A \cos 3x + B \sin 3x) - 8 \sin 3x$

$= -9A \cos 3x - 12B \cos 3x + 13A \cos 3x - 9B \sin 3x + 12A$
$\sin 3x + 13B \sin 3x - 8 \sin 3x$

$= (-9A - 12B + 13A) \cos 3x + (-9B + 12A + 13B) \sin 3x = 8 \sin 3x$

$= (4A - 12B) \cos 3x + (4B + 12A) \sin 3x = 8 \sin 3x$

Comparing Co-efficents..

$\sin 3x \Rightarrow \quad 4B + 12A = 8 \rightarrow (a)$

$\cos 3x \Rightarrow \quad 4A - 12B = 0 \quad 4A = 12B$

$\boxed{A = 3B}$ — (b)

Put (b) in eq (a)

$4B + 12(3B) = 8$

$4B + 36B = 8$

$40B = 8$

$\boxed{B = 1/5} \longrightarrow (c)$ Put eq (c) in eq (b)

$\boxed{A = 3/5} \rightarrow (d)$

Put eq (c) and (d) in (x)

$y_p = \frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x$ — (B)

The G. Solution is

$y = y_c + y_p$

$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{3}{5} \cos 3x$
$+ \frac{1}{5} \sin 3x \rightarrow (1)$

Scanned with CamScanner

Now we need to find the values of C_1 and C_2 for this.

Put $x=0$ and $y=1$ in eq (C)

$$1 = e^{2(0)}\left(c_1 \cos 3(0) + c_2 \sin 3(0)\right) + \frac{3}{5}\cos 3(0) + \frac{1}{5}\sin 3(0)$$

$$1 = \left(c_1 (1) + c_2 (0)\right) + \frac{3}{5}(1) + \frac{1}{5}(0)$$

$$1 = C_1 + \frac{3}{5}$$

$$C_1 = 1 - \frac{3}{5} \Rightarrow \boxed{C_1 = 2/5} \rightarrow (\text{XA})$$

Diff (C) w.r.to "x"

$$y' = c_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + c_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) - \frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x \quad \text{— (D)}$$

Put $y' = 2$ $x=0$ in (D)

$$y' = c_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + c_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) - \frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x$$

Put $y'=2$, $x=0$

$$2 = c_1\left(2e^{(0)}\cos 3(0) - 3e^{2(0)}\sin(3(0))\right) + c_2\left(2e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0)\right) - \frac{6}{5}\sin 3(0) + \frac{3}{5}\cos 3(0)$$

$$2 = c_1(2) + c_2(3) - 0 + \frac{3}{5}$$

$$2 = 2c_1 + 3c_2 + \frac{3}{5}$$

Put $c_1 = 2/5$

$$2 = \frac{4}{5} + 3c_2 + 3/5$$

$$2 = \frac{7}{5} + 3c_2$$

$$3c_2 = 2 - \frac{7}{5} \Rightarrow 3c_2 = 3/5 \Rightarrow \boxed{c_2 = \frac{3}{15}}$$

(XB)

Put XA and XB in eq C

$$y = e^{2x}\left(\frac{2}{5}\cos 3x + \frac{3}{15}\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$$

$$\boxed{y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x}$$

Required general solution.

Q No # 4

Solve $(D^2 - DD')z = \cos x \cos 2y$

Solution:-

It already in symbolic from

$$(D^2 - DD')z = \cos x \cos 2y \longrightarrow (1)$$

As we know

$\frac{D}{D'} = m$ i.e. $D = m$, $D' = 1$

$\Rightarrow m^2 - m = 0$

$m = 0, 1$

Therefore

$C.F = f_1(y) + f_2(y+x)$

From eq (a)

$$PI = \frac{1}{D^2 - DD'} \cos x \cos 2y$$

$$= \frac{1}{2} \frac{1}{D^2 - DD'} 2\cos x \cos 2y$$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$CF = f_1(y-x) + x f_2(y-x)$

$$PS = \frac{1}{D^2 - 2DD' + D'^2} \left[2(y-x) + \sin(x-y)\right]$$

$$= \frac{1}{(D+D')^2} \left[2(y-x) + \sin(x-y)\right]$$

By General Method:-

$m_2 = -1$; $y - x = c$

$$= \frac{1}{D+D'}\left[(2c+\sin(-c)\right]dx$$

$$= \frac{1}{D+D'}\left[2cx-(\sin c)x\right]$$

$$= \frac{1}{D+D'}\left[2cx-(\sin c)x\right]$$

Replacing c by y-x

$$= \frac{1}{D+D'}\left[2x(y-x)-x\sin(y-x)\right]$$

Again put $y-x=c$

$$\int(2xc - x\sin c)\,dx \Rightarrow \left(cx^2 - \frac{x^2}{2}\sin c\right)$$

Replacing 'c' by $y-x$

$$x^2(y-x) - \frac{x^2}{2}\sin(y-x) =$$

$$= x^2y - x^3 + \frac{x^2}{2}\sin(x-y)$$

Hence

The Required Solution is

$$z = C.F + PI = f_1(y-x) + x f_2(y-x) + x^2y - x^3 + \frac{1}{2}x^2\sin(x-y)$$