Differential Equation

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Date :- 30/6/2020

Q No 01 (part-a)

Ans:- Given data

As given Symmetry of wave is expressed by one-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

Where
w = Wave height
x = distance variable
t = time variable
c = Velocity

Now, to show that

$$w = \sin(x+ct) + \cos(2x+2ct)$$

is a solution of wave equation we need to find partial derivative.

Solution:-

$$W = \sin(x+ct) + \cos(2x+2ct)$$

partial derivative

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial t}\left(\sin(x+ct) + \cos(2x+2ct)\right)$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) + c - \sin(2x+2ct) + 2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) + c^2 - \cos(2x+2ct) + 4c^2 \longrightarrow (a)$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) - \sin(2x+2ct) + 2$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$= +c^2\left[-\sin(x+ct) - 4\cos(x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial t^2} = +c^2\left[-\sin(x+ct) - 4\cos(2x+2ct)\right]$$

$$\boxed{c^2 * \frac{\partial^2 w}{\partial x^2} \quad \text{or} \quad c^2 \cdot \frac{\partial^2 w}{\partial x^2}}$$

Ans.

Q No 01

(part – b)

Ans:: **Given data**

$$W = \tan(2x + ct)$$

Required:- To check if it the solution of given eq or not.

Solution:- $W = \tan(2x + ct)$

partial diff w.r.t "x"

Now

$$\frac{\partial w}{\partial t} = c\sec^2(2x + ct)$$

again Diff;

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c\sec^2(2x+ct)\right)$$

$$= 2c^2 \cdot 2\sec^2(2x+ct)\tan(2x+ct)$$

$$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

and $$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

$$\Rightarrow 4c^2\sec^2(2x+ct)\tan(2x+ct)$$

$$\Rightarrow 4c^2\sec^2(2x+ct)\tan(2x+ct)$$

$$\Rightarrow 0 = 0$$

satasified ;

Hence
$W = \tan(2x+ct)$ is

the solution of given eq

Q No 02

Ans :- Given data

$F(x) = x, \quad -\pi < x \leq 0$

$2x ; \quad 0 \leq x \leq \pi$

Solution

$$F(x) = x \quad -\pi < x \leq 0$$
$$= 2x \quad 0 \leq x \leq \pi$$

Expanding in a fourier Series

$$f(x) = \left\{ \begin{array}{ll} x; & -\pi < x \leq 0 \\ 2x; & 0 \leq x \leq \pi \end{array} \right]$$

We need to find farior co-efficient

a_0, a_n, b_n

$$a_0 = \frac{1}{x} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{x} \int_{\pi}^{0} x\, dx + \frac{1}{x} \int_{0}^{\pi} 2x\, dx$$

$$\Rightarrow \frac{1}{\pi} \left(\frac{\pi^2}{2} \right]_{-\pi}^{0} + \frac{2}{\pi} \left(\frac{\pi^2}{2} \right|_{0}^{\pi}$$

$$= -\frac{1}{\pi}\left(0 - \frac{\pi^2}{2}\right) + \frac{2}{\pi}\left(\frac{\pi^2}{2} - 0\right)$$

$$\Rightarrow a_0 = -\frac{\pi}{2} + \pi$$

$$\boxed{a_0 = \frac{\pi}{2}} \longrightarrow (1)$$

a_n;
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\cos nx \, dx$$

$$\Rightarrow \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos nx}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos nx}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

$$a_n = \begin{cases} \dfrac{-2}{\pi n^2} \; ; & \text{if } n \text{ is odd} \\ 0 \; ; & \text{if } n \text{ is even} \end{cases}$$ —— (2)

Now

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}\left(x\left(-\frac{\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right)_{0}^{\pi}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left(\frac{-\pi\cos n\pi}{n}\right) + \frac{2}{\pi}\left(\frac{-\pi\cos n\pi}{n}\right)$$

$$\Rightarrow b_n = -\frac{3\cos n\pi}{n}$$

$$\Rightarrow \boxed{b_n = \frac{3(-1)^{n+1}}{n}}$$ —— (3)

Hence from eq 1, 2 and 3

The Required fourier Series is

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\boxed{\Rightarrow \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{2n-1} + 3\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}}$$

Ans

QNo 03

Ans:- **Given data**

$$y'' - 4y' + 13y = 8\sin 3x$$

$$y(0) = 1$$

$$y'(0) = 2 .$$

Solution:- $y'' - 4y' + 13y = 8\sin 3x \longrightarrow (1)$

Associated Homogenous Eq of (1) is

$$y'' - 4y' + 13y = 0 \longrightarrow (2)$$

Change eq (2) into Auxiliary equation
put $y = m$ in (2)

$$m^2 - 4m + 13 = 0$$

Use Quadratic formula

$a = 1$, $b = 4$ $c = 13$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(Quadratic formula)}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16-52}}{2} \Rightarrow \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36}i}{2} \qquad \therefore i = \sqrt{-1}$$

$$\Rightarrow \frac{4 \pm 6i}{2}$$

$$m = 2 \pm 3i$$

$$\boxed{\begin{aligned} m &= 2+3i \\ m &= 2-3i \end{aligned}}$$

$$y_c = e^{2x}(c_1\cos 3x + c_2 \sin 3x) \longrightarrow (A)$$

$$y_p = A\cos 3x + B\sin 3x \longrightarrow (\star)$$

Differentiat w.r.t x

$$y_p' = -3A\sin 3x + 3B\cos 3x$$

Again Diff w.r.t x

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

put in eq (1)

$$\Rightarrow -9A\cos 3x - 12B\cos 3x + 13A + 13A\cos 3x - 9B\sin 3x$$
$$+ 12A\sin 3x + 13B\sin 3x - 8\sin 3x$$

$$\Rightarrow (-9A - 12B + 13A)\cos 3x + (-9B + 12A + 13B)\sin 3x$$
$$= 8\sin 3x$$

$$\Rightarrow (4A - 12B)\cos 3x + (4B + 12A)\sin 3x = 8\sin 3x$$

By Comparing Co-efficient

$\sin 3x \Rightarrow \quad 4B + 12A = 8 \quad$ — (a)

$4A - 12B = 0$

$$4A = 12B$$

$$\boxed{A = 3B} \rightarrow (b)$$

putting the value of A in eq (a)

$$4B + 12(3B) = 8$$

$$4B + 36B = 8$$

$$40B = 8$$

$$\boxed{B = \frac{1}{5}} \longrightarrow (c)$$

putting the value of B in eq (b)

$$A = 3B$$

$$A = 3(1/5)$$

$$\boxed{A = 3/5} \rightarrow (d)$$

put eq (c) & (d) in eq (*)

$$y_p = \frac{3}{5}\cos 3x + \frac{1}{3}\sin 3x \longrightarrow (B)$$

General solution is

$$y = y_c + y_p$$

$$y = e^{2x}(C_1\cos 3x + C_2\sin 3x) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \longrightarrow (C)$$

Now to find the value of "C_1" and "C_2"

given that $y(0) = 1$

put $x = 0$, $y = 1$ in eq (C)

$$1 = e^{2(0)}\left(C_1\cos 3(0) + C_2\sin 3(0)\right) + \frac{3}{5}\cos 3(0) + \frac{1}{5}(\sin 3(0))$$

$$1 = (C_1(1) + C_2(0)) + \frac{3}{5}(1) + \frac{1}{5}(0)$$

$$1 = C_1 + \frac{3}{5}$$

$$C_1 = 1 - \frac{3}{5} \quad \boxed{C_1 = \frac{2}{5}} \rightarrow (**)$$

Diff w.r.t "x"

$$y' = C_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + C_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) - \frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x \longrightarrow (D)$$

put $y' = 2$, $x = 0$ in eq (D)

$$y' = C_1\left(2e^{2x}\cos 3x - 3e^{2x}\sin 3x\right) + C_2\left(2e^{2x}\sin 3x + 3e^{2x}\cos 3x\right) - \frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x$$

put $y' = 2$, $x = 0$

$$2 = C_1\left(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0)\right) + C_2\left(2e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0)\right) - \frac{6}{5}\sin 3(0) + \frac{3}{5}\cos 3(0)$$

$$2 = C_1(2) + C_2(3) - 0 + \frac{3}{5}$$

$$2 = 2C_1 + 3C_2 + \frac{3}{5}$$

put $C_1 = \frac{2}{5}$

$$2 = \frac{4}{5} + 3C_2 + \frac{3}{5}$$

$$2 = \frac{7}{5} + 3C_2$$

$$3C_2 = 2 - \frac{7}{5}$$

$$\boxed{C_2 = \frac{3}{15}} \rightarrow ★★★$$

putting eq ** and *** in C

$$y = e^{2x}\left(\frac{2}{5}\cos 3x + \frac{3}{15}\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x.$$

$$\boxed{y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x}$$

Required General solution.

QNo 04

Ans:- Given data

$$(D^2 - DD')z = \cos x \cos 2y$$

Solution

$$(D^2 - DD')z = \cos x \cos 2y \quad \text{--- ⓐ}$$

As it is already in Symbolic form

$$(D^2 - DD')z = \cos x \cos 2y$$

put $A.E = D^2 - DD' = 0$ [A.E = Auxiliary Equation]

As we know

$$\frac{D}{D'} = m \quad \text{i.e } D = m, \; D' = 1$$

$$\Rightarrow m^2 - m = 0$$

$$m = 0, 1$$

Therefore $C.F = F_1(y) + f_2(y+x)$

From eq ⓐ

$$P.I = \frac{1}{D^2 - DD'} \cos x \cos 2y$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 - DD'} 2\cos x \cos 2y$$

As

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$C.F = f_1(y-x) + x f_2(y-x) \quad \left[C.F \Rightarrow \text{Complimentry Function} \right.$$

$$P.I = \frac{1}{D^2 + 2DD' + D'^2} \left[2(y-x) + \sin(x-y) \right]$$

$$= \frac{1}{(D+D')^2} \left[2(y-x) + \sin(x-y) \right]$$

By General Method

$$m = -1 \; ; \; y - x = c$$

$$= \frac{1}{D+D'} \left[\int [2c + \sin(-c)] \right] dx$$

$$= \frac{1}{D+D'} \left[2cx - (\sin c) x \right]$$

Replacing c by $y-x$

$$= \frac{1}{D+D'}\left[2x(y-x) - x\sin(y-x)\right]$$

put Again $y-x=c$

$$= \int (2xc - x\sin c)\,dx \Rightarrow cx^2 - \frac{x^2}{2}\sin c$$

Replacing c by $y-x$

$$\Rightarrow x^2(y-x) - \frac{x^2}{2}\sin(y-x) = x^2y - x^3 + \frac{x^2}{2}\sin(x-y)$$

The required solution is

$$Z = C.F + P.I = f_1(y-x) + xf_2(y-x) + x^2y - x^3 + \frac{1}{2}x^2\sin(x-y)$$