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SECTION B
SUBJECT DE
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ASSIGNMENT

QUESTION # 01

$$x^3y''' + 2x^2y' + 2y - 10x + 10/x$$

SOLUTION :~

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$$

$$x^3 \Delta^3 y + 2x^2 \Delta^2 + 2y = 10x + 10x^{-1}$$

$$(x^3\Delta^3 + 2x^2\Delta + 2)y = 10x + 10x^{-1} \quad (i)$$

let $x = e^t \Rightarrow t = \ln x$

$$x\Delta = \Delta$$

$$x^2\Delta^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$$

$$x^3 D^3 = \Delta(\Delta-1)(\Delta-2)$$

Subtituting into eq (i)

$$(\Delta - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2)y = 10^x + 10x^{-1}$$

$$(\Delta^3 - \Delta^2 + 2)y = 10x + 10x^{-1}$$

$$(m^3 - m^2 + 2)y = 10e^t + \frac{10}{e^t}$$

Using synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & -1 & 0 & 2 \\ & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$D^2 - 2D + 2 = 0$

Now using Quadratic Equation

$a = 1, \quad b = -2, \quad c = 2$

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a}$$

$D = 2 \pm \sqrt{4-8}/2$

$D = 2 \pm \sqrt{-4}/2$

$D = 2 \pm \sqrt{-1}\sqrt{4}/2$

$$D = \frac{2 + 2i}{2}$$

$$D = \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$D = 1 \pm i$

Since roots are complex

$y_c = e^{-x}(C_1 \cos t + C_2 \sin t)$

Now particular integration.

~~$y_p = 1/D^3 - D^2 + 2 \; 10e^t$~~

$$y_p = \frac{1}{D^3 - D^2 + 2} 10e^t + \frac{1}{D^3 - D^2 + 2} \cdot 10/e^t$$

$$= \frac{10e^t}{(1)^3 - (1)^2 + 2} + \frac{10e^{-t}}{(1)^3 - (1)^2 + 2}$$

$$= \frac{\cancel{10}^{5} e^t}{\cancel{2}} + \frac{\cancel{10}^{5} e^{-1}}{\cancel{2}}$$

$y_p = 5e^{t} + 5e^{t}$

General Solution :-

$y = y_c + y_p$

$y = e^{-x}(c_1 \cos t + c_2 \sin t) + 5e^{t} + 5e^{t}$

put $e^{t} = x$ and $t = \ln x$

$y = e^{-x}(c_1 \ln x + c_2 \sin \ln x) + 5e^{x} + 5e^{-1}$ Ans

QUESTION # 02

$$x^3 \frac{d^3y}{dx} + 4x^2 \frac{d^2y}{dx} - 5x \frac{dy}{dx} - 15y = x^4$$

SOLUTION :-

let $\frac{d}{dx} = D$

$x^3 D^3 y + 4x^2 D^2 y - 5x D y - 15y = x^4$

$(x^3 D^3 + 4x^2 D^2 - 5x D - 15) y = x^4$

let $x = e^{t} \Rightarrow t = \ln x$

$x D = D$

$x^2 D^2 = D(D-1) = D^2 - D$

$x^3 D^3 = D(D-1)(D-2) = D^3 - 3D^2 + 2D$

Now Substituting

$(x^3 D^3 + 4x^2 D^2 - 5x D - 15) y = x^4$

$D^3 - 3D^2 + 2D + 4(D^2 - D) - 5(D) - 15) y = e^{4t}$

$(D^3 + D^2 - 7D - 15) y = e^{4t}$

Synthetic division

	1	+1	-7	-15
5		3	12	15
	1	4	5	0

$$D^2 + 4D + 15 = 0$$

Quadratic Formula :-

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$$y_c = e^{3x}(c_1 \cos t + c_2 \sin t)$$

For $y_p = ?$

$$y_p = \frac{1}{D^3 + D^2 - 7D - 18} \cdot e^{4T}$$

$$= \frac{1}{(4)^3 + (4)^2 - 7(4) - 15} e^{4t}$$

$$= \frac{1}{64 + 16 - 28 - 15} e^{4t}$$

$$= \frac{1}{80 - 43} e^{4t}$$

$$y_p = \frac{1}{37} e^{4t}$$

Hence

$$y = y_c + y_p$$

$$y = c_1 \cos t + c_2 \sin t) + \frac{1}{37} e^{4t} \quad \therefore \text{Put } t = \ln x,\ x = \ln x$$

$$y = e^{3x}(c_1 \cos \ln x + c_2 \sin (\log x)) + \frac{1}{37} e^{4x} \quad \text{Ans}$$

QUESTION # 03

$$x^2 y'' + 2xy' - 6x = 10x^2$$

SOLUTION :~

$y(1) = 1$ & $y'(1) = -6$

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6x = 10x^2$$

$$\Rightarrow \left(x^2 \frac{d^2}{dx^2} + 2x\frac{d}{dx} - 6\right)y = 10x^2$$

put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x = e^t$ and $\log x = t$

$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{3t}$

$(\Delta^2 + \Delta - 6)y = 10e^{2t}$

$(\Delta^2 + \Delta - 6)y = 10e^{2t}$

The characteristic equation

$\Delta^2 + \Delta - 6 = 0$

$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$

$\Rightarrow \Delta(\Delta + 3) - 2(\Delta + 3) = 0$

$\Rightarrow (\Delta + 3)(\Delta - 2) = 0$

$\Delta + 3 = 0$, $\Delta - 2 = 0$

$\Delta = 2$, $\Delta = -3$

Since roots are real and distinct

For $y_c = ?$

$y_c = C_1 e^{-3t} + C_2 e^{2t}$

For $y_p = ?$

$$y_p = \frac{1}{\Delta^2 - \Delta - 6} \cdot 10^{2t}$$

$$= \frac{10}{\Delta^2 - \Delta - 6} e^{2t}$$

Now

$$10 \frac{1}{d/dD(D^2+D-6)} e^{2t}$$

$$\Rightarrow 10 \frac{t}{2D+1} e^{2t}$$

$$10 \frac{1 \cdot t}{4+1} e^{2t}$$

$y_p = 2te^{2t}$

GENERAL SOLUTION

$$y = y_c + y_p$$

$$= C_1 e^{-3t} + C_2 e^{2t} + 2te^{2t}$$

$$y = C_1 x^{-3} + C_2 x^2 + 2(\log x)\, x^2 \quad — (B)$$

put $y(1) = 1$ i.e $x = 1$, $y = 1$ in (B)

$$1 = C_1(1)^{-3} + C_2(1)^2 + 2\log(1)$$

$$1 = C_1 + C_2 \quad — (C)$$

Now differentiate eq (B) w.r.t x.

$$y' = -3C_1 x^{-4} + 2C_2 x + \frac{2}{x}(x^2) + 4x\log x$$

Now put $y'(1) = -6$ i.e $y' = -6$ and $x = -6$.

$$-6 = -3C_1 + 2C_2 + 2 + 0$$

$$\Rightarrow -6 = -3C_1 + 2C_2 + 2$$

$$-6 - 2 = -3C_1 + 2C_2 + 2$$

$$-8 = -3C_1 + 2C_2 \quad — (D)$$

$\times$'ing eq (C) with (2) & $\div$'ing from (D)

$$2C_1 + 2C_2 = 2$$

$$\mp 3C_1 \pm 2C_2 = \mp 8$$

$$5C_1 = 10$$

$$C_1 = {}^{2}\cancel{10}/\cancel{5} \qquad \boxed{C_1 = 2}$$

$$-8 = -3(2) + 2C_2 \Rightarrow -8 = -6 + 2C_2$$

$$2C_2 = -8 + 6 \Rightarrow 2C_2 = -2$$

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$2C_2 = -2$

$C_2 = -\cancel{2}/\cancel{2}\ 1$

$\boxed{C_2 = -1}$

Now put the value of C_1 & C_2 in eq (B)

$y = 2x^{-3} - x^2 + 2\ln x\ x(x^2)$

$$y = \frac{2}{x^3} - x^2 + 2x^2 \log x \quad \text{ANS}$$

QUESTION # 04

$$x^2y'' + 7xy' + 5y = x^5$$

SOLUTION:-

$y(0) = 2$ & $y'(1) = 2$

$x^2 \frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 5y = x^5$

$\Rightarrow (x^2 \frac{d^2}{dx^2} + 7x\frac{d}{dx} + \cancel{5y}\,5)y = x^5$ —(A)

put $xD = D \Rightarrow x^2D^2 = D(D-1) = D^2 - D$

$x = e^t \Rightarrow \log x = t$ in eq (A)

$\Rightarrow (D^2 - D + 7D + 5)y = e^{5t}$

$\Rightarrow (D^2 + 6D + 5)y = e^{5t}$

By Quadratic Formula

$D = b \pm \sqrt{b^2 - 4ac}$

$D = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$ $2a$

$D = \frac{-6 \pm \sqrt{36 - 20}}{2}$

$D = \frac{-6 \pm \sqrt{16}}{2}$

$D = \frac{-6 \pm 4}{2} \Rightarrow D = \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}}$

$D = -3 \pm 2$

Since roots are real and distinct

$y_c = C_1 e^{-5t} + C_2 e^{-t}$

For $y_p = ?$

$$y_p = \frac{1}{D^2+6D+5} e^{5t}$$

$$= \frac{1}{(5)^2+6(5)+5} e^{5t}$$

$$= \frac{1}{16} \cdot e^{5t}$$

Now General Solution is

$y = y_c = y_p$

$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t}$

$y = C_1 x^{-5} + C_2 x^{-1} + \frac{1}{60} x^5 \quad —(B)$

$x = 0$ put in this equation

No in eq (B) $e^0 = 1$

put in $y(0) = 2$ i.e $y = 2$ & $x = 2$

$2 = C_1 (2)^{-5} + C_2 (2)^{-1} + \frac{1}{60}(2)^5$

$2 = -32C_1 - 2C_2 \; \frac{1}{\cancel{60}_{15}}(\cancel{32}^{8})$

$2 = -32C_1 - 2C_2 + \frac{8}{15}$

$2 - \frac{8}{15} = -32C_1 - 2C_2$

$\frac{22}{15} = -32C_1 - 2C_2 \quad —(C)$

Now differentiate eq(B) w.r.t (x)

$y' = -5C_1 x^{-6} - C_2 x^{-2} + \frac{1}{2} x^4 \longrightarrow (ii)$

put $y'(1) = 2$ i.e $y' = 2$ & $x = 2$ In eq (ii)

$2 = -5C_1 \; C_2^{-6} - C_2(2)^{-2} + \frac{1}{12}(2)^4$

$2 = -5C_1(-64) - C_2(4) + \frac{1}{12}(16)$

$2 = 320C_1 + 4C_2 + \frac{4}{3}$

$\Rightarrow 2 - \frac{4}{3} = 320C_1 + 4C_2$

$\frac{2}{3} = 320C_1 + 4C_2 \quad —(D)$

×ing eq (C) with 2 and then −ing eq (C) from (D)

$-\frac{44}{15} = 64C_1 + 4C_2$

$-\frac{44}{15} = 64C_1 + \cancel{4C_1}$

$\underline{\pm \frac{2}{3} = \pm 320C_1 \pm 4C_2}$

$\frac{34}{15} = 256C_i$

$C_1 = \frac{34}{15} \times 256$

$C_1 = 580$

put the value of C_1 in eq (c)

$\frac{22}{15} = -32(580) - 2C_2$

$\frac{22}{15} = -18560 - 2C_2$

$\frac{22}{15} + 18560 = -2C_2$

$\frac{18561}{-2} = C_2$

$-9280 = C_2$

Now put the value of C_1 & C_2 in eq (B)

$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x5$

$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$

Ans

QUESTION # 05

$$(x+1)^2 y'' - 3(x+1)' y' + 4y = x^2$$

Solution :~

$$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$\Rightarrow \left((x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right)y = x^2$$

$$\Rightarrow \left((x+1)^2 \Delta^2 - 3(x+1)\Delta + 4\right)y = x^2 \quad \text{—(A)}$$

put $(x+1)\Delta = \Delta \Rightarrow (x+1)^2\Delta^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$x = e^t$ in eq (A)

$$\Rightarrow (\Delta^2 - \Delta - 3\Delta + 4)y = e^{2t}$$

$$\Rightarrow \Delta^2 - 4\Delta + 4)y = e^{2t}$$

$$\Rightarrow \Delta^2 - 4\Delta + 4)d = e^{2t}$$

for y_c we find the roots

$$\Delta^2 - 4\Delta + 4 = 0$$

$$\Delta^2 - 2\Delta - 2\Delta + 4 = 0$$

$$\Delta(\Delta-2) - 2(\Delta-2) = 0$$

$\Delta - 2 = 0$, $\Delta = 2$

$\Delta - 2 = 0$, $\Delta = 2$

So roots are real and repeat the General solution

$$y = (C_1 + C_2 x)^{2x}$$

$$y = (C_1 + C_2 x)^{2t}$$

For $y_p = ?$

$$y_p = \frac{1}{\Delta^2 - 4\Delta + 4}$$

$$y_p = \frac{2}{2\Delta - 4} e^{2t}$$

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if we put 2

$2D - 4 \Rightarrow 2(2) - 4 = 0$

we take again derivation

$y_p = \frac{x}{\cancel{2}} \cdot e^{2t}$

$y = (C_1 + C_2 x)^{2t} + e^{2t}$