Final Term Assignment
Multivariate Calculus

Time Allowed: 6 hours ***Marks: 50***

Note: Attempt all questions. Copying from Internet and one another is strictly prohibited. Such answers will be marked zero.

Q1.

Find $\dfrac{\partial^2 z}{\partial x \partial y}$ and $\dfrac{\partial^2 z}{\partial y \partial x}$ for $z = arc\sin\left(\dfrac{x}{y}\right)$

Q2

Show that the function $f(x, y) = e^x \sin y + e^y \cos x$ satisfies the Laplace's equation.

Q3

If $f(x, y) = x^3 e^{-y} + y^3 \sec x$, then find the partial derivatives of *f(x, y)* with respect to *x* and *y*.

Q4

Find the vector projection of $b = 6i + 3j + 2k$ onto $a = i - 2j - 2k$.

Q5

Find the directional derivative of $f(x,y)=xe^y + \cos(xy)$ at the point $(2,0)$in the direction of $a = 3\hat{i} - 4\hat{j}$.

Q6

Find the Equation of the tangent plane and normal of the surface f(x,y,z)= $x^2 + y^2 + z^2 - 14$, at the point P(1,-2,3)

Q7.

Evaluate the double integral $\int_0^1 \int_0^1 (xy + y^2)\,dx\,dy$

Q8.

Find the area of the region R enclosed by the parabola $y = x^2$ and the line

Q1:- Find $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ for $z = \text{arc} \sin\left[\frac{x}{y}\right]$

<u>Answers.</u>

Since arc is constant with respect to x,

Differentiate using the chain rule, which states that $\frac{d}{dx}[f(g(x))]$ is $f'(g(x))g'(x)$ where $f(x) = \sin(x)$ and $g(x) = \frac{x}{y}$.

To apply the chain Rule, set u as $\frac{x}{y}$.

$$\text{arc}\left(\frac{d}{du}[\sin(u)]\frac{d}{dx}\left[\frac{x}{y}\right]\right.$$

The derivative of $\sin(u)$ with respect to u is $\cos(u)$.

$$\text{arc}\left(\cos(u)\frac{d}{dx}\left[\frac{x}{y}\right]\right)$$

Replace all occurrences of u with $\frac{x}{y}$.

$$\text{arc}\left(\cos\left(\frac{x}{y}\right)\frac{d}{dx}\left[\frac{x}{y}\right]\right)$$

Differentiate.

Since $\frac{1}{y}$ is constant with respect to x, the derivative of $\frac{x}{y}$ with respect to x is $\frac{1}{y}\frac{d}{dx}[x]$.

$$\text{arc}\cos\left(\frac{x}{y}\right)\left(\frac{1}{y}\frac{d}{dx}[x]\right)$$

Combine a and $\frac{1}{y}$.

$$\arccos\left(\frac{x}{y}\right)\left(\frac{a}{y}\frac{d}{dx}[x]\right).$$

Combine r and $\frac{a}{y}$.

$$c\cos\left(\frac{x}{y}\right)\left(\frac{ra}{y}\frac{d}{dx}[x]\right)$$

Combine $\frac{ra}{y}$ and c.

$$\frac{rac}{y}\cos\left(\frac{x}{y}\right)\frac{d}{dx}[x]$$

Combine $\frac{rac}{y}$ and $\cos\left(\frac{x}{y}\right)$.

$$\frac{rac\cos\left(\frac{x}{y}\right)}{y}\frac{d}{dx}[x]$$

Differentiate using the power Rule which states that $\frac{d}{dx}[x^n]$ is nx^{n-1} where $n=1$.

$$\frac{rac\cos\left(\frac{x}{y}\right)}{y}\cdot 1$$

Simplify the expression.

Multiply $\frac{rac\cos\left(\frac{x}{y}\right)}{y}$ by 1.

$$\frac{rac\cos\left(\frac{x}{y}\right)}{y}$$

Reorder terms.

$$arc\cos\left(\frac{x}{y}\right)$$

Q2:-

Show that the function $f(x,y) = e^x \sin y + e^y \cos x$ satisfy the Laplace's equation.

Answer:-

Differentiate both sides of the equation:

$$\frac{d}{dy}(f(x,y)) = \frac{d}{dy}(e^x \sin(y) + e^y \cos(x))$$

Since f is constant with respect to y, the derivative of $f(x,y)$ with respect to y is $f \frac{d}{dy}[f(x,y)]$. $f \frac{d}{dy}[(x,y)]$

Differentiate the right side of the equation

By the sum Rule, the derivatives of $e^x \sin(y) + e^y \cos(x)$ with respect to y is $\frac{d}{dy}[e^x \sin(y)] + \frac{d}{dy}[e^y \cos(x)]$.

$$\frac{d}{dy}[e^y \cos(x)]$$

$$e^x \cos(y) + \sin(y) e^x \frac{d}{dy}[x] + \frac{d}{dy}[e^y \cos(x)]$$

$$e^x \cos(y) + \sin(y) e^x \frac{d}{dy}[x] + e^y(-\sin(x) \frac{d}{dy}[x] + \cos(x) \frac{d}{dy}[e^y]$$

Rewrite $\frac{d}{dy}[x]$ as $\frac{d}{dy}[x]$. $e^x \cos(y) + \sin(y)$

$$e^x \frac{d}{dy}[x] + e^y(-\sin(x) \frac{d}{dy}[x] + \cos(x) \frac{d}{dy}[e^y]$$

Differentiate using the Exponential Rule which states that $\frac{d}{dy}[a^y]$ is $a^y \ln(a)$ where $a = e$.

$$e^x \cos(y) + \sin(y) e^x \frac{d}{dy}[x] + e^y(-\sin(x)\frac{d}{dy}[x]$$
$$+ e^y(-\sin(x)\frac{d}{dy}[x]) + \cos(x) e^y$$

Move -1 to the left of e^y.

$$e^x \cos(y) + \sin(y) e^x \frac{d}{dy}[x] - 1$$
$$\cdot e^y \sin(x) \frac{d}{dy}[x] + \cos(x) e^y$$

Reorder term.

$$e^x \cos(y) + e^x \sin(y) \frac{d}{dy}[x)$$
$$- e^y \sin(x) \frac{d}{dy}[x] + e^y \cos(x)$$

Reform the equation by setting the left side equal to the right side.

$$f\frac{d}{dy}((x,y)) = e^x \cos(y) + e^x \sin(y) x' - e^y$$
$$\sin(x) x' + e^y \cos(x)$$

Solve for x

$$e^x \cos(y) + e^x \sin(y) x' - e^y \sin(x) x'$$
$$+ e^y \cos(x), e^x \cos(y) + x' e^x \sin(y) -$$
$$x' e^y \sin(x) + e^y \cos(x) = \frac{fd}{dy((x,y))}$$

Ans

Q4:- Find the vector projection of $b = 6i+3j+2k$ onto $a = i-2j-2k$.

Answer:-

$\vec{e} = 6\hat{i}+3\hat{j}+2\hat{k}$

$\vec{f} = \hat{i}-2\hat{j}-2\hat{k}$

$|\vec{e}| = \sqrt{6^2+3^2+2^2} = 11$

$|\vec{f}| = \sqrt{1^2-2^2-2^2} = -1$

$$| = \frac{\vec{e}\cdot\vec{f}}{|\vec{f}|} \quad \text{—— (1)}$$

$$= \frac{\vec{e}\cdot\vec{f}}{(|\vec{f}|)^2}(\vec{f})$$

$$Eq(2) \Rightarrow \frac{\vec{e}\cdot\vec{f}}{(|\vec{f}|)^2}(\vec{f}) = \frac{[(6\hat{i}+3\hat{j}+2\hat{k})\cdot\hat{i}-2\hat{j}-2\hat{k}]}{(-1)^2}(\hat{i}-2\hat{j}$$

$$= \frac{[(6)(1)+(3)(2)+(2)(2)]}{1}(\hat{i}-2\hat{j}-2\hat{k})$$

vector projection

$$\vec{e} \text{ on } \vec{f} = \frac{16}{9}(\hat{i}+2\hat{j}-2\hat{k})$$

$$= \frac{16}{11}i + \frac{32}{1}j - \frac{32}{1}k$$

Ans

Q6:-

Find the equation of the tangent plane and normal of the Surface $f(x,y,z) = x^2+y^2+z^2-14$, at the point $P(1,-2,3)$

$\Rightarrow f_x(x,y,z) = 2x+0+0-14 = 2x-14$

$\Rightarrow f_y(x,y,z) = 0+2y+0-14 = 2y-14$

$\Rightarrow f_z(x,y,z) = 0+0+2z-14 = 2z-14$

Now putting points.

$\Rightarrow f_x(1,-2,3) = 2(1)-14 = 2-14 = -12$

$\Rightarrow f_y(1,-2,3) = 2(-2)-14 = -4-14 = -16$

$\Rightarrow f_z(1,-2,3) = 2(3)-14 = 6-14 = -8$

Now with equation of tangent plane.

$\Rightarrow f_x(x_0,y_0,z_0)(x-x_0) +$

$f_y(x_0,y_0,z_0)(y-y_0) +$

$f_z(x_0,y_0,z_0)(z-z_0)$

$\Rightarrow -12(x-1)-16(y+2)-8(z-4)$

$\Rightarrow -12x+12-16y-3z-16+24$

$\Rightarrow -12x-16y-3z-20=0$

Ans

(7)

Q7. Evaluate the double integral $\int_0^1\int_0^1 (xy+y^2)\,dx\,dy$.

double integral

$$\int_0^1\int_0^1 (xy+y^2)\,dy\,dx$$

→ first we will solve inner integral

$$= \int_0^1 (xy+y^2)\,dy$$

$$= \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3}\right]$$

$$= \frac{x}{2}\left((1)^2-(0)^2\right)+\frac{1}{3}\left((1)^3-(0)^3\right)$$

$$= \frac{x}{2}(1)+\frac{1}{3}(1)$$

$$= \frac{x}{2}+\frac{1}{3} \text{ or } \frac{3x+2}{6}$$

Now $\int_0^1 \left(\frac{x}{2}+\frac{1}{3}\right)dx$.

$$\int_0^1 \frac{x^2}{2\times 2} + \frac{1}{3} = \frac{1}{4}\left((1)^2-(0)^2\right)+\frac{1}{3}$$

$$\frac{1}{4}+\frac{1}{3} = \frac{3+4}{12} = \frac{7}{12} \text{ Ans}$$

Q 8.

Find the area of the region R enclosed by the parabola $y=x^2$ and the line $y=x+2$.

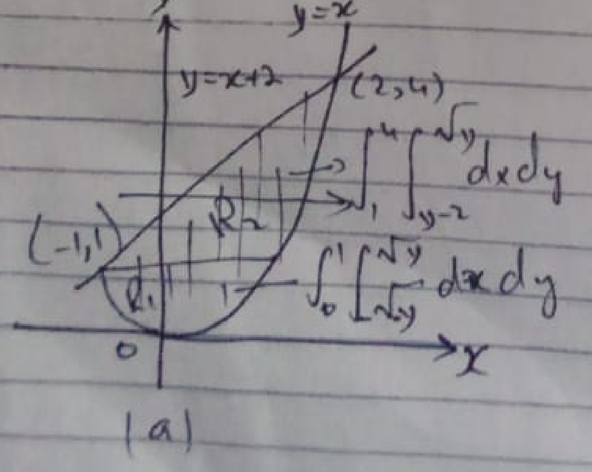
Answer:

If we divid R into the regions R_1 and R_2. We may calculate the area as

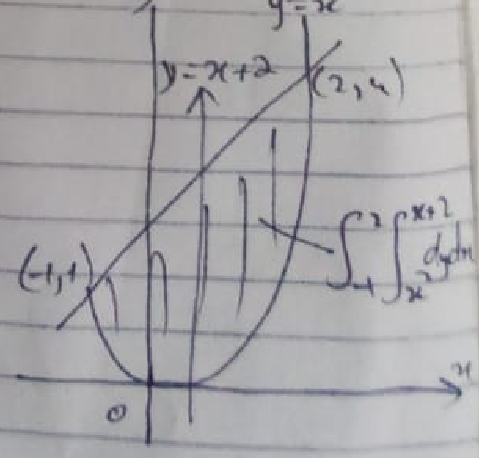
$$A=\iint_{R_1} dA+\iint_{R_2} dA=\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy+\int_1^4\int_{y-2}^{\sqrt{y}} dx\,dy$$

On the other hand, reversing the order of integration

$$A=\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx.$$



(a)



(b)

Calculate this area takes (a) two double integrals if the first integration is with respect to x, but (b) only if the first integration is with respect to y

This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. the area is

$$A = \int_{-1}^{2} \left[y \right]_{x^2}^{x+2} dx = \int_{-1}^{2} (x+2-x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}$$

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f\,dA.$$

Ans:-

Question no 3:-
If $f(x,y) = x^3 e^{-y} + y^3 \sec x$, then find the partial derivatives of $f(x,y)$ with respect to x and y.

Answer:-

By the sum rule, the derivative of $x^3 e^{-y} + y^3 \sec(x)$ with respect to x is $\frac{d}{dx}[x^3 e^{-y}] + \frac{d}{dx}[y^3 \sec(x)]$.

$$\frac{d}{dx}[x^3 e^{-y}] + \frac{d}{dx}[y^3 \sec(x)]$$

Evaluate $\frac{d}{dx}[x^3 e^{-y}]$.

Since e^{-y} is constant with respect to x, the derivative of $x^3 e^{-y}$ with respect to x is $e^{-y} \frac{d}{dx}[x^3]$.

$$e^{-y} \frac{d}{dx}[x^3] + \frac{d}{dx}[y^3 \sec(x)]$$

Differentiate using the Power Rule which states that $\frac{d}{dx}[x^n]$ is nx^{n-1} where $n=3$.

$$e^{-y}(3x^2) + \frac{d}{dx}[y^3 \sec(x)]$$

move 3 to the left of e^{-y}.

$$3e^{-y}x^2 + \frac{d}{dx}[y^3 \sec(x)]$$

Evaluate $\frac{d}{dx}[y^3 \sec(x)]$.

Since y^3 is constant with respect to x, the derivative of $y^3 \sec(x)$ with respect to x is $y^3 \frac{d}{dx}[\sec(x)]$.

$3e^{-y}x^2 + y^3 \frac{d}{dx}[\sec(x)]$

The derivative of $\sec(x)$ with respect to x is $\sec(x)\tan(x)$.

$3e^{-y}x^2 + y^3 \sec(x)\tan(x)$

Ans

Question no 05:-
Find the directional derivatives of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $a = 3\hat{i} - 4\hat{j}$.

Answer:

$\frac{d}{dx}[f(x)\, e(y)]$ is $f(x)\frac{d}{dx}[e(x)] + e(x)\frac{d}{dx}[f(x)]$ where $f(x) = x$ and $g(x) = \ln(x)$. $x\frac{d}{dx}[\ln(x)] + \ln(x)\frac{d}{dx}[x]$

x is xe^y.

$x\frac{1}{x} + \ln(3\hat{i} - 4\hat{i})$.

$\frac{d}{dx}[x^n]$ is nx^{n-1}

where $n = 1$.

$xe^y + \ln(x) \cdot 1$

Combine

x and $xe^y + \cos(xy)$

at points

2.0

M.