Iqra National University, Peshawar

Department of Computer Science

Spring Semester, Date: 25th June 2020

Final term – Semester Examination

Course Code: **Course Title: Differential Equations**

Instructor: Engr. Latif Jan

Program: BS (CS-SE & EE) **Total Marks: 50** **Time Allowed: 120**

minutes Note: Attempt all Questions:

Q 1: a) Define 2nd order linear homogenous/non-homogenous differential equation along with examples? **(1+1 Marks)**

b) Solve the following 2nd order Linear homogeneous /non-homogenous differential equation? **(5+5 Marks)**

i. 4y''-6y'+7y=0
ii. y''-
4y'12y=3e^(5x)

Q 2: Solve the following IVP for the 2nd order linear equations. **(5+5+5+5 Marks)**

(i) 16y′′−40y′+25y=0 y(0)=3 y′(0)=−9/4
(ii) y′′+14y′+49y=0 y(−4)=−1y′(−4)=5
(iii) y′′−4y′+9y=0 y(0)=0y′(0)=−8
(iv) y′′−8y′+17y=0 y(0)=−4y′(0)=−1

Q 3: Define Laplace transform along with example? **(2 Marks)**

A. Find the Laplace transforms of the given functions. **(2+2+2 Marks)**

1. f(t) = 6(e^−5t)+e^3t+5(t^3)−9
2. g(t) = 4cos(4t)−9sin(4t)+2cos(10t)
3. h(t) = e^3t+cos(6t)−e^(3t)cos(6t)

Q4: Solve the following IVP using Laplace Transform. **(5+5 Marks)**

(i) $y'' - 10y' + 9y = 5t$, $\quad y(0) = -1, \quad y'(0) = 2$

(ii) $y'' - 6y' + 15y = 2\sin(3t)$, $\quad y(0) = -1 \quad y'(0) = -4$

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Q.1.a. Define 2nd order linear homogenous/non-homogenous differential equation along with examples?

Second-Order Homogeneous Equations:-

There are two definitions of the term "homogeneous differential equation." One definition calls a first-order equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Homogeneous if *M* and *N* are both homogeneous functions of the same degree. The second definition — and the one which you'll see much more often—states that a differential equation (of *any* order) is **homogeneous** if once all the terms involving the unknown function are collected together on one side of the equation, the other side is identically zero. For example,

$$y'' - 2y' + y = 0 \quad \text{is homogeneous}$$

But

$$y'' - 2y' + y = x \quad \text{is not}$$

The nonhomogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = d(x) \qquad (*)$$

can be turned into a homogeneous one simply by replacing the right-hand side by 0:

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad (**)$$

Second-Order Non-Homogeneous Equations:-

The nonhomogeneous differential equation of this type has the form

$y''+py'+qy=f(x)$,
Where p,q are constant numbers (that can be both as real as complex numbers). For each equation we can write the related homogeneous or complementary equation:

$y''+py'+qy=0$.
The non-homogenous differential equation has terms on both side in this type of equation has the form of

$$a(x)y'' + b(x)y' + c(x)y = d(x) \qquad (*)$$

Q1. b i) $4y'' - 6y' + 7y = 0$

Sol

$$\Rightarrow 7y(x) - 6\frac{d}{dx}y(x) + 4\frac{d^2}{dx^2}y(x) = 0$$

÷ing whole equation by 4 ∟

$$\Rightarrow \frac{7y(x)}{4} - \frac{3\frac{d}{dx}y(x)}{2} + \frac{d^2}{dx^2}y(x) = 0$$

where

$$p = -\frac{3}{2}$$

$$q = \frac{7}{4}$$

This is called linear homogeneous 2nd order differential eq

To find Roots ∟

Equation $\Rightarrow q + (k^2 + kp) = 0$

$$\Rightarrow k^2 - \frac{3k}{2} + \frac{7}{4} = 0$$

Roots of the equation:

$$k_1 = \frac{3}{4} - \frac{\sqrt{19}i}{4}$$

$$k_2 = \frac{3}{4} + \frac{\sqrt{19}i}{4}$$

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eq 2) $y(x) = e^{k_1 x} C_1 + e^{k_2 x} C_2$

$$y(x) = C_1 e^{x\left(\frac{3}{4} - \frac{\sqrt{19}i}{4}\right)} + C_2 e^{x\left(\frac{3}{4} + \frac{\sqrt{19}i}{4}\right)}$$

Q1.b.

ii) $y'' - 4y' - 12 = 3e^{5x}$

Sol

Given Equation is

$$-12y(x) - 4\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = 3e^{5x}$$

The differential Equation has the form:

$$y'' + py' + qy = s,$$

where

$$p = -4$$
$$q = -12$$
$$s = -3e^{5x}$$

It is called linear inhomogeneous second order differential equation with const. coefficients

$$\Rightarrow \quad q + (k^2 + kp) = 0$$

$$Eq = \quad k^2 - 4k - 12 = 0$$

The roots of this equation.

$$k_1 = -2$$
$$k_2 = 6$$

As there are two roots of the characteristics equation, and the roots are not complex, then

$$y(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$

$$y(x) = C_1 e^{-2x} + C_2 e^{6x}$$

Now we use variation of parameters method

$$y(x) = C_1(x) e^{-2x} + C_2(x) e^{6x}$$

$$\Rightarrow \frac{d}{dx} C_1(x) \frac{d}{dx} y_1(x) + \frac{d}{dx} C_2(x) \frac{d}{dx} y_2(x)$$

where

$y_1(x)$ & $y_2(x) \Rightarrow$ linearly independent.

$$\Rightarrow y_1(x) = \exp(-2x) \quad (C_1 = 1, C_2 = 0)$$

$$y_2(x) = \exp(6x) \quad (C_1 = 0, C_2 = 1)$$

The free term $f = -S$ or

$$f(x) = 3e^{5x}$$

$$\Rightarrow e^{6x} \frac{d}{dx} C_2(x) + e^{-2x} \frac{d}{dx} C_1(x) = 0$$

$$\Rightarrow \frac{d}{dx} C_1(x) \frac{d}{dx} e^{-2x} + \frac{d}{dx} C_2(x) \frac{d}{dx} e^{6x} = 3e^{5x}$$

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or $e^{6x}\frac{d}{dx}C_2(x) + e^{-2x}\frac{d}{dx}C_1(x) = 0$

$$6e^{6x}\frac{d}{dx}C_2(x) - 2e^{-2x}\frac{d}{dx}C_1(x) = 3e^{5x}$$

Solving the System:

$$\frac{d}{dx}C_1(x) = -\frac{3e^{7x}}{8}$$

$$\frac{d}{dx}C_2(x) = \frac{3e^{-x}}{8}$$

It is the simple differential equation.

Now solving it:

$$C_1(x) = C_3 + \int\left(-\frac{3e^{7x}}{8}\right)dx$$

$$C_2(x) = C_4 + \int\frac{3e^{-x}}{8}\,dx$$

or

$$C_1(x) = C_3 = -\frac{3e^{7x}}{56}$$

$$C_2(x) = C_4 = -\frac{3e^{-x}}{8}$$

Substitute found $C_1(x)$ & $C_2(x)$ to

$$y(x) = C_1(x)e^{-2x} + C_2(x)e^{6x}$$

$$\Rightarrow y(x) = C_3 e^{-2x} + C_4 e^{6x} - \frac{3e^{5x}}{7}$$

Where C_3 & C_4 is a Constant.

Q2 i) $16y'' - 40y' + 25y = 0 \quad y(0) = 3 \quad y'(0) = -9/4$

Sol

Given equation:

$$16y'' - 40y' + 25y = 0$$

$\div$ing whole equation by 16:

$$\frac{25y(x)}{16} - \frac{5\frac{d}{dx}y(x)}{2} + \frac{d^2}{dx^2}y(x) = 0$$

where

$$p = -\frac{5}{2}$$

$$q = \frac{25}{16}$$

To find Roots:

$$\Rightarrow q + (k^2 + kp) = 0$$

Equation $= k^2 - \frac{5k}{2} + \frac{25}{16} = 0$

Root $=$ $k_1 = \frac{5}{4}$

$$\Rightarrow y(x) = e^{k_1 x}C_1 + e^{k_1 x}C_2 x$$

Substitute $\Rightarrow k_1 = \frac{5}{4}$

$$\Rightarrow y(x) = C_1 e^{\frac{5x}{4}} + C_2 x e^{\frac{5x}{4}}$$

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Q8 ii) $y'' + 14y' + 49y = 0 \quad y(-4) = -1, \; y'(-4) = 5)$

Sol

$$49y(x) + 14\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = 0$$

where

$p = 14$

$q = 49$

To find roots:

$$q + (k^2 + kp) = 0$$

Equation = $\quad k^2 + 14k + 49 = 0$

Root: $\quad k_1 = -7$

$$\Rightarrow y(x) = e^{k_1 x} C_1 + e^{k_1 x} C_2 x$$

Substitute

$k_1 = 7$

$$\Rightarrow y(x) = C_1 e^{-7x} + C_2 x e^{-7x}$$

Solution of Cauchy Problem

$y(-4) = 1$

$$\left(\begin{cases} -4 & \text{for } 0 = 1 \\ 1 & \text{for } 1 = 1 \\ 0 & \text{otherwise} \end{cases}\right) \frac{d}{dx} y(x) \Bigg|_{x=-4} = 5$$

$$\frac{d}{dx} y(x) = C_2 e^{-7x} - 7(C_1 + C_2 x)e^{-7x}$$

$$y(x) = (C_1 + C_2 x)e^{-7x}$$

$$5 = C_2 e^{--28} - 7(C_1 + (-4)C_2)e^{--28}$$

$$\Rightarrow -1 = (C_1 + (-4)C_2)e^{--28}$$

$$C_1 = -\frac{9}{e^{28}}$$

$$C_2 = -\frac{2}{e^{-28}}$$

$$y(x) = \left(\frac{-2x}{e^{28}} - \frac{9}{e^{28}}\right)e^{-7x}$$

$$\Rightarrow y(x) = (C_1 + C_2 x)e^{-7x}.$$

Q2 iii) $y'' - 4y' + 9y = 0 \qquad y(0) = 0 \quad y'(0) = 8.$

Sol

$$\Rightarrow 9y(x) - 4\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = 0$$

where

$$P = -4$$
$$q = 9$$

To find Roots

$$q(k^2 + kp) = 0$$

Equation =

$$k^2 - 4k + 9 = 0$$

Roots

$$k_1 = 2 - \sqrt{5}i$$
$$k_2 = 2 + \sqrt{5}i$$

$$\Rightarrow y(x) = e^{k_1 x}C_1 + e^{k_2 x}C_2$$

$$\Rightarrow y(x) = C_1 e^{x(2-\sqrt{5}i)} + C_2 e^{x(2+\sqrt{5}i)}$$

The Solution Cauchy Problem =

$$y(0) = 0$$

$$\left(\begin{cases} 0 & \text{for } 0 = 1 \\ 1 & \text{for } 1 = 1 \\ 0 & \text{otherwise} \end{cases} \right) \frac{d}{dx} y(x) \Bigg|_{x=0} = -8$$

$$\frac{d}{dx} y(x) = 2(c_1 \sin\sqrt{5}x) + c_2 \cos(\sqrt{5}x))e^{2x}$$

$$y(x) = (c_1 \sin\sqrt{5}x) + c_2 \cos(\sqrt{5}x))e^{2x}$$

$$-8 = 2(c_1 \sin(0\sqrt{5}) + c_2 \cos(0\sqrt{5}))e^{0.2} + (\sqrt{5})$$

$$0 = (c_1 \sin(0\sqrt{5}) + c_2 \cos(0\sqrt{5}))e^{0.2}$$

$$c_2 = 0$$

$$c_1 = \frac{-8\sqrt{5}}{5}$$

$$y(x) = \frac{-8\sqrt{5}e^{2x} \sin(\sqrt{5}x)}{5}$$

Q2 iv) $y'' - 8y' + 17y = 0$ $\quad y(0) = -4 \quad y'(0) = 1$

Sol:

$$17y(x) - 8\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = 0$$

where

$$p = -8$$
$$q = 17$$

To find Roots:

$$q + (k^2 + kp) = 0$$

Equation

$$k^2 - 8k + 17 = 0$$

Roots:

$$k_1 = 4 - i$$
$$k_2 = 4 + i$$

$$\Rightarrow y(x) = e^{k_1 x} C_1 + e^{k_2 x} C_2$$

$$\Rightarrow y(x) = C_1 e^{x(4-i)} + C_2 e^{x(4+i)}$$

$$\Rightarrow y(x) = (C_1 \sin(x) + C_2 \cos(x))e^{4x}$$

Q 3: **Define Laplace transform along with example?**

Laplace transformation:-

It is a technique for solving differential equations. Here differential equation of time domain form is first transformed to algebraic equation of frequency domain form. After solving the algebraic equation in frequency domain, the result then is finally transformed to time domain form to achieve the ultimate solution of the differential equation. In other words it can be said that the Laplace transformation is nothing but a shortcut method of solving differential equation.

In this article, we will be discussing Laplace transforms and how they are used to solve differential equations. They also provide a method to form a transfer function for an input-output system, but this shall not be discussed here. They provide the basic building blocks for control engineering

Then the Laplace transform of $f(t), F(s)$ can be defined as:

$$F(s) = \int_0^\infty f(t).e^{-st} dt$$

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Q3 A. Find the laplace transforms of the given function.

1) $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

Sol

$$\Rightarrow f(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

2) $g(t) = 4\cos^{4t} - 9\sin^{4t} + 2\cos^{(10t)}$

Sol

$$\Rightarrow g(s) = 4\frac{s}{s^2+4^2} - 9\frac{4}{s^2+4^2} + 2\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

3) $h(t) = e^{3t} + \cos^{6(t)} - e^{3t}\cos^{6t}.$

Sol

$$g(s) = \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

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Q4 :) $y'' - 10y' + 9y = 5t, \quad y(0) = -1 \; y'(0) = 2$

Sol

Applying Laplace transform to both side,

$\Rightarrow (s^2 - 10s + 9)Y - s - 2 - 10 = \frac{5}{s^2} \Rightarrow Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$

Applying ~~Fraction~~ Partial Fraction,

$$\frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

We find

$$B = +\frac{5}{9}, \quad D = -2, \quad C = \frac{31}{81}, \quad A = \frac{50}{81}$$

Therefore, using linearity of the inverse Laplace function.

$$y(t) = \frac{50}{81} + \frac{5t}{9} + \frac{31}{81}e^{9t} - 2e^{t}$$

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Q4 ii) $y'' - 6y' + 15y = 2\sin 3(t)$ $y(0) = -1$ $y'(0) = -4$

Sol

We have

$$(s^2-6s+15)Y + s - 2 - \frac{6}{s^2+9} \cancel{-Y(s)} \Rightarrow \cancel{-s^2+2s^2} - 9$$

$$\Rightarrow Y(s) = \frac{-s^2+2s^2-9s+24}{(s^2+9)(s^2-6s+15)}$$

$$= \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}$$

Finding Co efficients

$\Rightarrow$

$s^3 = A + C = 1$

$s^2 = -6A + B - D = 2$

$s' = 15A - 6B + K = -9$

$s^0 = 15B + 9D - 24$

$$\Rightarrow A = \frac{1}{10}, \; B = \frac{1}{10}, \; C = \frac{-11}{10}, \; D = \frac{5}{2}$$

$$\Rightarrow Y(s) = \frac{1}{10}\left(\frac{s-1}{s^2+9} + \frac{-11s+25}{s^2-6x+15}\right)$$

Now find inverse Laplace transform.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \cos 3t + \frac{1}{3}\sin 3t$$

$$\Rightarrow \frac{-11s+25}{s^2-6s+15} = \frac{-11s+25}{(s-3)^2+6}$$

$$= \frac{-11-(s-3)-8}{(s-3)^2-6}.$$

$$= -11\frac{(s-3)}{(s-3)^2+6} - \frac{8}{\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}$$

Now

$$\mathcal{L}^{-1}\left\{\frac{-11s+25}{s^2-6s+15}\right\} = -11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}t$$

Therefore

$$y(t) = \mathcal{L}^{-1}\{Y\} = \frac{1}{10}\left(\cos 3t + \frac{1}{3}\sin 3t - 11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t} - \sin\sqrt{6}\,6\right)$$