Final Term Exam

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Subject: Differential Equation

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Q1 a) Define 2^{nd} . . . .

Ans: Second order linear equation homogenous differential equation is defined as: (with constant coefficients:-

$$y'' + ay' + by = 0 \text{ (a \& b are numbers)}$$

e.g: $3y'' - 6y' + 2y = 0$

Note: There is no term that is based on function of x itself

In a non-homogenous D.E:-
there is a term made of only the function of x.

$$y'' + Py' + P_0 y = Q(x)$$

e.g: $3y'' - 6y + 2y = e^{5t}$

b) Solve the following 2^{nd} order . . . . .

i. $4y'' - 6y' + 7y = 0$

Solution:

→ Second order homogenous equation:
→ Making the characteristic equation:-
→ Using undeterminant coefficient

$$4m^2 - 6m + 7 = 0$$

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→ Using quadratic equation:

$$m = \frac{6 \pm \sqrt{36 - 4(4)(7)}}{2(4)}$$

$$m = \frac{3 \pm i\sqrt{19}}{4} \Rightarrow \frac{3}{4} \pm i\frac{\sqrt{19}}{4}$$

→ Hence:

$$y = c_1 e^{\frac{3x}{4}} \cos\left(\frac{\sqrt{19}x}{4}\right) + c_2 e^{\frac{3}{4}x} \sin\left(\frac{\sqrt{19}}{4}x\right)$$

$$y(x) = e^{\frac{3}{4}x}\left[c_1 \cos\left(\frac{\sqrt{19}x}{4}\right) + c_2 \sin\left(\frac{\sqrt{19}x}{4}\right)\right]$$ Ans

ii $y'' - 4y' - 12y = 3e^{5x}$

Solution:

→ Homogeneous equation:

$$y'' - 4y' - 12y = 3e^{5x}$$

→ Characteristic equation:

$$m^2 - 4m - 12 = 0$$

$$m = 6, -2$$

→ Hence complementary solution is:

$$y_c = c_1 e^{6x} + c_1 e^{-2x}$$

→ Finding particular solution:

→ Let $y_p = ae^{5x}$

$$y_p' = 5ae^{5x} \quad ; \quad y'' = 25ae^{5x}$$

→ Substituting in DE:

$$25ae^{5x}-4(5ae^{5x})-12ae^{5x}=3e^{5x}$$
$$(25a-20a-12a)e^{5x}=3e^{5x}$$
$$(-7a)e^{5x}=3e^{5x}$$
$$-7a=3 \Rightarrow a=\frac{-3}{7}$$

→ Hence
$$y_p=\frac{-3}{7}e^{5x}$$

→ So, since $y(x)=y_c+y_p$

$$y(x)=c_1e^{5x}+c_2e^{-2x}-\frac{3}{7}e^{5x}$$ Ans

Q2 Solve the following IVP for the ....

(i) $16y''-40y'+25=0 \quad y(0)=3 \; y'(0)=-9/4$

Solution:

$$16m^2-40m+25=0 \text{ (Characteristic equation)}$$
$$m=\frac{5}{4}, \frac{5}{4}$$

→ Hence, $y(x)=(c_1+c_2x)e^{\frac{5}{4}x}$

$y(0)=3$
$$y(0)=[c_1+c_2(0)]e^{\frac{5}{4}(0)}$$
$$3=c_1$$
$$y'(x)=\frac{5}{4}e^{\frac{5}{4}x}(c_1+c_2x)+(c_2)e^{\frac{5}{4}x}$$
$$y'(0)=\frac{5}{4}e^0(c_1+0)+(c_2)e^0$$

$$-\frac{9}{4} = \frac{5}{4}(3) + c_2 \Rightarrow c_2 = -6$$

→ Hence:

$$\underline{y(x) = e^{\frac{5}{4}x}(3-6x)}$$ Ans

ii $y'' + 14y' + 49y = 0 ;\ y(-4) = -1 \quad y'(-4) = 5$

Solution:

→ Characteristic equation:

$$m^2 + 14m + 49 = 0$$

$$m = -7, -7$$

$$y(x) = e^{-7x}(c_1 + c_2 x)$$

$$y(-4) = e^{28}(c_1 - 4c_2)$$

$$-1 = e^{28}(c_1 - 4c_2) \Rightarrow c_1 - 4c_2 = \frac{-1}{e^{28}} \quad \text{— (i)}$$

$$y'(x) = -7e^{-7x}(c_1 + c_2 x) + (c_2)e^{-7x}$$

$$y'(-4) = -7e^{28}(c_1 + c_2(-4)) + c_2 e^{28}$$

$$5 = e^{28}\left[-7c_1 + 28c_2 + c_2\right]$$

$$-7c_1 + 29c_2 = \frac{5}{e^{28}} \quad \text{— (ii)}$$

→ Solving (i) & (ii) simultaneously

$$c_1 = 4c_2 - \frac{1}{e^{28}}$$

$$-7(4c_2 - e^{-28}) + 29c_2 = 5e^{-28}$$

$$-28c_2 + 7e^{-28} + 29c_2 = 5e^{-28}$$

$$c_2 = 5e^{-28} - 7e^{-28}$$

$$C_2 = -2e^{-28}$$
$$C_1 = 4(-2e^{-28}) - e^{-28}$$
$$C_1 = -9e^{-28}$$

→ Hence:

$$y(x) = e^{-7x}\left[-9e^{-28} + (-2e^{-28})x\right]$$

$$y(x) = -9e^{-28-7x} - 2e^{-28-7x}x$$

$$y(x) = -e^{-28-7x}(+9+2x) \quad \text{Ans}$$

iii $y'' - 4y' + 9y = 0 \quad y(0) = 0, \; y'(0) = -8$

Solution:

→ Characteristic equation:

$$m^2 - 4m + 9 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4(1)(9)}}{2(1)}$$

$$m = 2 \pm \sqrt{5}\,i$$

$$y(x) = e^{2x}(c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x))$$

$$y(0) = e^{0}(c_1 \cos(0) + c_2 \cancel{\sin(0)})$$

$$0 = c_1$$

$$y(x) = e^{2x}(c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x))$$

$$y'(x) = 2e^{2x}(c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x)) +$$
$$(c_1 \sin(\sqrt{5}x)(\sqrt{5}) + (\sqrt{5}c_2)\cos\sqrt{5}x))$$

$$y'(0) = 2e^{0}(c_2 \sin(0) + \sqrt{5}c_2 \cos(0))$$

$$-8 = \sqrt{5}c_2 \Rightarrow c_2 = \frac{-8}{\sqrt{5}}$$

→Hence:

$$y(x) = e^{2x}\left(\frac{-8}{\sqrt{5}}\right)\sin(\sqrt{5}x)$$

$$y(x) = \frac{-8e^{2x}}{\sqrt{5}}\sin(\sqrt{5}x) \quad \text{Ans}$$

(iv) $y'' - 8y' + 17y = 0$; $y(0) = -4$, $y'(0) = -1$

Solution:

$$m^2 - 8m + 17 = 0 \text{ (characteristic eq)}$$

$$m = \frac{8 \pm \sqrt{64 - 4(1)(17)}}{2(1)}$$

$$m = 4 \pm i$$

$$y(x) = e^{4x}(c_1 \cos x + c_2 \sin x)$$

$$y(0) = e^{0}(c_1 + 0)$$

$$-4 = c_1$$

$$y'(x) = 4e^{4x}(c_1 \cos x + c_2 \sin x) + (-c_1 \sin x + c_2 \cos x)e^{4x}$$

$$y'(0) = 4(-4+0) + (1)(c_2)$$

$$-1 = -16 + c_2 \Rightarrow c_2 = 15$$

→Hence:

$$y(x) = e^{4x}(15\sin x - 4\cos x) \quad \text{Ans}$$

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Q3 Define Laplace transform along with example?

→ Let $f(t)$ be a given function of t defined for all $t \geq 0$ then the Laplace Transform of $f(t)$ denoted by $L\{f(t)\}$ of $F(s)$ is defined as.

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

provided the integral exists, where s is a parameter real or complex

e.g: Let $f(t) = 1$, then Laplace Transform of $f(t)$ will be

$$L(f(t)) = \int_0^\infty (1)\, e^{-st}\, dt$$

$$" = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$" = \left[0 - \left(\frac{-1}{s}\right)\right] = \frac{1}{s}$$

So, $F(s) = \frac{1}{s}$

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A. Find the Laplace transform of the given functions.

1. $f(t) = 6(e\text{\^{}}{-5t}) + e\text{\^{}}3t + 5(t\text{\^{}}3) - 9$

Solution:

→ Note: $L[e^{kt}] = \frac{1}{s-k}$

$L[t^n] = \frac{n!}{s^{n+1}}$

→ Hence:

$$L[f(t)] = 6\left[\frac{1}{s+6}\right] + \left[\frac{1}{s-3}\right] + 5\left[\frac{6}{s^4}\right] - \frac{9}{s}$$

$$L[f(t)] = \frac{6}{s+6} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

2. $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

Solution:

→ Note: $L\left[\cos at\right] = \frac{s}{s^2+a^2}$

$L\left[\sin at\right] = \frac{a}{s^2+a^2}$

$$\to L[g(t)] = 4\left[\frac{s}{s^2+16}\right] - 9\left[\frac{4}{s^2+16}\right] + 2\left[\frac{s}{s^2+100}\right]$$

$$L[g(t)] = \frac{4s+36}{s^2+16} + \frac{2s}{s^2+100}$$ **Ans**

iii $h(t) = e\hat{}3t + \cos(6t) - e\hat{}(3t)\cos(6t)$

Solution:

→ Note: Using first shift theorem

$$L(e^{-kt}\sin at) = \frac{a}{(s+k)^2+a^2}$$

$$L(e^{-kt}\cos at) = \frac{s+k}{(s+k)^2+a^2}$$

→ Hence

$$L(h(t)) = \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{(s-3)}{(s-3)^2+36}$$

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Q4 Solve the following IVP using Laplace Transform.

(i) $y'' - 10y' + 9y = 5t \qquad y(0) = -1, \; y'(0) = 2$

Solution:

→ Homogeneous differential equation.

$$y'' - 10y' + 9y = 0$$

→ Characteristic equation.

$$m^2 - 10m + 9 = 0$$
$$m = 9, 1$$

→ (Complimentary solution) $y_c(t) = c_1 e^{9t} + c_2 e^{t}$

→ Finding particular solution:

$$y_p = at + b$$
$$y_p' = a \;, \Rightarrow y_p'' = 0$$

$$(0) - 10(a) + 9(at + b) = 5t$$
$$-10a + 9at + 9b = 5t$$
$$9at + (9b - 10a) = 5t$$
$$9a = 5 \;; \; 9b - 10a = 0$$
$$a = \frac{5}{9} \Rightarrow b = \frac{10}{9}\left(\frac{5}{9}\right) \Rightarrow b = \frac{50}{81}$$
$$y_p = \frac{5t}{9} + \frac{50}{81}$$

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→ Hence,

$$y(t) = y_c + y_p$$

$$y(t) = c_1e^{9t} + c_2e^{t} + \frac{5}{9}t + \frac{50}{81}$$

$$y(0) = c_1(1) + c_2(1) + \frac{50}{81}$$

$$-1 = c_1 + c_2 + \frac{50}{81} \Rightarrow c_1 + c_2 = \frac{-131}{81} \quad \text{—(i)}$$

$$y'(t) = 9c_1e^{9t} + c_1e^{t} + \frac{5}{9}$$

$$y'(0) = 9c_1 + c_2 + \frac{5}{9}$$

$$9c_1 + c_2 = \frac{13}{9} \quad \text{—(ii)}$$

→ Solving (i) & (ii) simultaneously:

$$c_1 = \frac{31}{81}, \quad c_2 = -2$$

→ Hence:

$$y(t) = \frac{31}{81}e^{9t} - 2e^{t} + \frac{5}{9}t + \frac{50}{81}$$

(ii) $y'' - 6y' + 15y = 2\sin 3t \,;\; y(0) = -1, y'(0) = 4$

Solution:

→ Characteristic equation:

$$m^2 - 6m + 15 = 0$$

$$m = \frac{6 \pm \sqrt{6^2 - 4(1)(15)}}{2(1)}$$

$$m = 3 \pm i\sqrt{6}$$

→ Hence:

$$y_c = e^{3t}(c_1 \cos\sqrt{6}t) + c_2 \sin(\sqrt{6}t))$$

(complimentary solution)

→ Finding particular solution:-

$$y_p = a\sin 3t + b\cos 3t$$

$$y_p' = 3a\cos 3t - 3b\sin 3t$$

$$y_p'' = -9a\sin 3t - 9b\cos 3t$$

$$(-9a + 18b + 15a)\sin 3t + (-9b - 18a + 15b)\cos 3t = 2\sin 3t$$

$$(6a + 18b)\sin 3t + (-18a + 6b)\cos 3t = 2\sin 3t$$

$$6a + 18b = 2 \longrightarrow (i)$$

$$-18a + 6b = 0 \longrightarrow (ii)$$

$$a = \frac{1}{30}, \quad b = \frac{1}{10}$$

→ Hence,

$$y_p = \frac{1}{30}\sin 3t + \frac{1}{10}\cos 3t$$

$$y(t) = e^{3t}\left[c_1\cos\sqrt{6}t + c_2\sin\sqrt{6}t\right] + \frac{1}{30}\sin 3t + \frac{1}{10}\cos 3t$$

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$$y(0) = (c_1) + \frac{1}{10}(1) \Rightarrow \quad c_1 = \frac{-11}{10}$$

$$y'(t) = e^{3t}\left[-\sqrt{6}c_1 \sin\sqrt{6}t + \sqrt{6}c_2 \cos\sqrt{6}t\right] + 3e^{3t}\left[c_1\cos\sqrt{6}t + c_2 \sin\sqrt{6}t\right] + \frac{1}{10}\cos 3t - \frac{3}{10}\sin 3t$$

$$y'(0) = (\sqrt{6}c_2) + 3(c_1) + \frac{1}{10}$$

$$-4 - \frac{1}{10} = \sqrt{6}c_2 + 3\left[\frac{-11}{10}\right]$$

$$c_2 = \frac{-2\sqrt{6}}{15}$$

Hence;

$$y(t) = e^{3t}\left[\frac{-11}{10}\cos\sqrt{6}t - \frac{2\sqrt{6}}{15}\sin\sqrt{6}t\right] + \frac{1}{30}\sin 3t + \frac{1}{10}\cos 3t$$

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