Mid Term Exam

ID: 11533

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Semester: 12^{th}

Subject: Differential Equation

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(a)
Q1 Define differential equation along with two examples:

Answer:
A differential equation is an equation which contains one or more terms which involve the derivatives of one variable (i-e: dependant variable) with respect to the other variable (i-e: independant variable)

$$\frac{dy}{dx} = f(x)$$

Here "x" is an independant variable and "y" is a dependant variable.

Examples:

① $y'' + 2y' = 3y$ or $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3y$

② $\frac{dy}{dx} = 5x$

(b)
Q1 Define a seperable Differential Equation.

Answer:
A seperable differential equation is a common kind of differential equation that is especially straight forward to solve. Seperable equations have the form $\frac{dy}{dx} = f(x)\,g(y)$

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x and y can be brought to opposite sides of the equation. Then, integrating both sides gives y as a function of x solving the differential equation.

Q! (i) Solve the following Initial value problem using seperable DE and find the interval of validity of the solution.

(a) $y' = \dfrac{xy^3}{\sqrt{1+x^2}}$ $\quad y(0) = -1$

Solution:-

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

→ By seperating we get

$$y^{-3}\frac{dy}{dx} = x(1+x^2)^{-\frac{1}{2}}$$

$$y^{-3}dy = x(1+x^2)^{-\frac{1}{2}}dx$$

→ Now by integration we get

$$\int y^{-3}dy = \int x(1+x^2)^{-\frac{1}{2}}dx$$

$$\frac{-1}{2y^2} = \int x(1+x^2)^{-\frac{1}{2}}dx$$

→ here

$$u = 1+x^2$$

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$$du = 2x\,dx$$

$$\frac{du}{2} = x\,dx \,,\quad \frac{du}{dx} = 2x$$

→ So

$$-\frac{1}{2}y^2 = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{2}$$

$$-\frac{1}{2}y^2 = \frac{1}{2}\int u^{-\frac{1}{2}} \cdot du$$

$$-\frac{1}{2}y^2 = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

→ Now by applying power rule

$$\int u^n dv = \frac{u^{n+1}}{n+1} \quad \text{with } n = \frac{-1}{2}$$

$$\Rightarrow 2\sqrt{u}$$

→ Now

$$\frac{-1}{2y^2} = \frac{1}{2}\int \frac{1}{\sqrt{v}}\,dv$$

$$-\frac{1}{2}y^2 = \sqrt{v}$$

where $v = x^2+1$ / $1+x^2$

→ so

$$-\frac{1}{2}y^2 = \sqrt{1+x^2} + C \longrightarrow (A)$$

→ Now putting the values $x=0, y=-1$ in equation (A), we get.

$$-\frac{1}{2}(-1)^2 = \sqrt{1+(0)^2} + C$$

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$$c = \frac{-1}{2} - \sqrt{1}$$

$$c = -\frac{3}{2}$$

→ Now putting the value of c in equation A to get the implicit solution.

$$\frac{-1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

→ Solving for $y(x)$

$$\frac{1}{y^2} = \sqrt{1+x^2} - \frac{3}{2} \times 2$$

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

→ Now taking squareroot on bothsides we get

$$y(x) = \pm \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

→ Reapplying the initial condition which shows "−" is the correct sign so, we get

$$y(x) = \frac{-1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

⇒ Now for interval =

Since $1x^2 \geq 0$

So $3 - 2\sqrt{1+x^2} > 0$

$3 > 2\sqrt{1+x^2}$

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→ Now taking square

$$(3)^2 > (2^2(\sqrt{1+x^2})^2)$$

$$9 > 4(1+x^2)$$

→ ÷ both sides by 4 we get

$$\frac{9}{4} > 1+x^2$$

→ Now subtracting 1 from b·s

$$\frac{9}{4} - 1 > x^2 + \cancel{1} - \cancel{1}$$

$$\frac{9-4}{4} > x^2$$

$$\frac{5}{4} > x^2$$

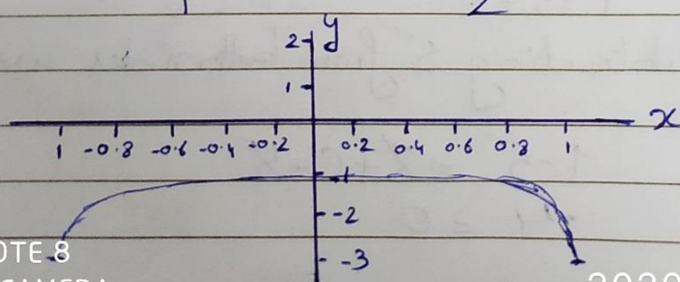
→ Now taking square root

$$\sqrt{\frac{5}{4}} > \sqrt{x^2}$$

$$\sqrt{\frac{5}{4}} > x$$

→ which means

$$-\sqrt{\frac{5}{4}} < x < \sqrt{\frac{5}{2}}$$



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Q1 b $y' = e^{-y}(2x-4)$, $y(5) = 0$

Solution:

$$y' = e^{-y}(2x-4), \quad y(5) = 0$$

$$y' = \frac{dy}{dx} = e^{-y}(2x-4)$$

→ Now by integrating both sides and seperating

$$\int e^{y}\,dy = \int (2x-4)\,dx$$

$$e^{y} = \frac{2x^2}{2} - 4(x) + C$$

$$e^{y} = x^2 - 4x + C \rightarrow \text{(A)}$$

→ integration of constant is x e.g $= \int 1 \Rightarrow 1(x)$

→ So,

$$e^{y} = x^2 - 4x + C$$

→ Now applying the given condition

$$e^{y} = (5)^2 - 4(5) + C$$

$$e^{y} = 25 - 20 + C$$

$$1 = 25 - 20 + C \qquad \because e^{y} = 1$$

→ Now

$$1 = 5 + C$$

→ subtracting 5 from bothsides we get

$$1 - 5 = \cancel{5} + C + \cancel{5}$$

$$-4 = C$$

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$c = -4$

→ Now from equation c, we get to find implicit solution.

$$e^y = x^2 - 4x + c$$

→ putting the value of c, we get

$$e^y = x^2 - 4x - 4$$

→ Now we will take natural log of both sides

$$\underline{y(x) = \ln(x^2 - 4x - 4)}$$

= For Interval of validity =

$$x^2 - 4x - 4 > 0$$

→ As we know that

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \longrightarrow B$$

→ where $b = -4$, $a = 1$ and $c = -4$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-4)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16+16}}{2}$$

$$x = \frac{\cancel{4}}{\cancel{2}} \pm \frac{4\sqrt{2}}{2}$$

$$x = 2 \pm 2\sqrt{2}$$

→ So

$x = 2 + 2\sqrt{2}\ ,\ 2 - 2\sqrt{2}$

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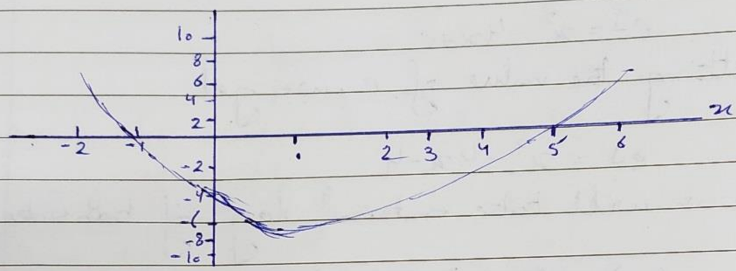
→ So we will get the positive value.
Quadratic will be zero at

$x = 2 \pm 2\sqrt{2}$

→ So possible interval validity will be.



: Quadratic graph:

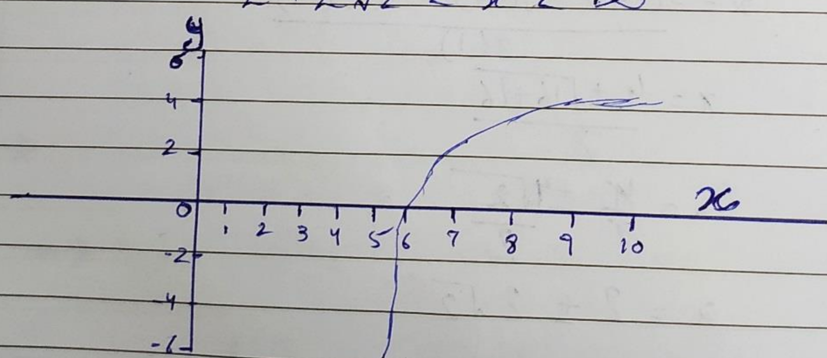
→ So possible interval of validity are

$-\infty < x < 2 - 2\sqrt{2}$

$2 + 2\sqrt{2} < x < \infty$

→ from the graph of the quadratic we can see the second one contains $x = 5$, So therefore

$2 + 2\sqrt{2} < x < \infty$



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Q2. Solve the following IVP using Linear differential method.

(i) Explain the steps for solving Linear Differential Equation.

STEPS:-

Step 1 First we put the differential equation in correct form (if its not)

$$\frac{dy}{dx} + p(x)y = g(x)$$

Step 2 Now we find the integrating factor $\mu(x)$, by using the below equation.

$$\mu(x) = e^{\int p(x)dx}$$

Step 3 Now we multiply everything in the differential equation by $\mu(x)$ and verify that the left side becomes the product rule.

$$(\mu(x)y(x))'$$

$$\mu(x)\frac{dy}{dx} + \mu'(x)y = (\mu(x)y(x))' \quad \text{— (A)}$$

Now as we know

$$\mu(t)\frac{dy}{dt} + \mu'(x)y = \mu(x)\,y(x)$$

So we will replace the leftside of the eq (A) with this, we get.

$$(\mu(x)y(x))' = \mu(x)\,g(x)$$

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Step 4 In fourth step we will integrate both sides and we will make sure to properly deal with constant.

$$\int (\mu(x)\,y(x))'\,dx = \int \mu(x)\,g(x)\,dx$$

$$\mu(x)\,y(x) + c = \int \mu(x)\,g(x)\,dx$$

Step 5: Now in last step, we will get $y(x)$

$$\mu(x)\,y(x) = \int \mu(x)\,g(x)\,dx - c$$

$$y(x) = \frac{\int \mu(x)\,g(x)\,dx - c}{\mu(x)}$$

(ii) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$,

$$y\left[\frac{\pi}{4}\right] = 3\sqrt{2}, \quad 0 \le x \le \frac{\pi}{2}$$

Solution:

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$$

→ Dividing by $\cos(x)$, we get

$$y' + \frac{\sin(x)(y)}{\cos(x)} = \frac{2\cos^{2}(x)\sin(x)}{\cancel{\cos(x)}} - \frac{1}{\cos x}$$

→ As we know $\therefore \frac{\sin}{\cos} = \tan$, $\frac{1}{\cos} = \sec$

→ So putting values.

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$y' + \tan(x)(y) = 2\cos^2(x)\sin(x) - \sec(x)$ — (A)

→ Integrating factors:

$\mu(x) = e^{\int \tan(x)\,dx} = e^{\ln|\sec(x)|} = e^{\ln \sec(x)}$

$= \sec(x)$

→ Now multiplying the integrating factor through the differential equation (A) and verifying the left side is a product rule.

$\sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$

$(\sec(x)y)' = 2\cos(x)\sin(x) - \sec^2(x)$

→ Now integrating both sides

$\int (\sec(x)y(x))'\,dx = \int 2\cos(x)\sin(x) - \sec^2(x)\,dx$

→ By integrating we get:

$\sec(x)\,y(x) = \int \sin(2x) - \sec^2(x)\,dx$

→ Now again integrating, we get.

$\sec(x)\,y(x) = \int \sin(2x) - \int \sec^2(x)\,dx$

$\sec(x) \cdot y(x) = \frac{1}{2}\int \sin(2x)\,dx - \tan(x)$

$\sec(x) \cdot y(x) = -\frac{\cos(2x)}{2} - \tan(x) + c$

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$\sec(x)\,y(x) = \frac{-1}{2}\cos(2x) - \tan(x) + C \rightarrow (K)$

→ Now multiplying $\cos(x)$ to the right side of (K) we get -

$y(x) = \frac{-1}{2}\cos(2x)\cos(x) - \cos(x)\tan(x) + C(\cos(x))$

$= \frac{-1}{2}\cos(x)\cos(2x) - \sin(x) + C\cos(x) \rightarrow$

→ Now applying the initial condition

$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = \frac{-1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(2\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) + C\cos\left(\frac{\pi}{4}\right)$

$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = \frac{-1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + C\cos\left(\frac{\pi}{4}\right)$

→ So

$$3\sqrt{2} = \frac{-\sqrt{2}}{2} + C\frac{\sqrt{2}}{2}$$

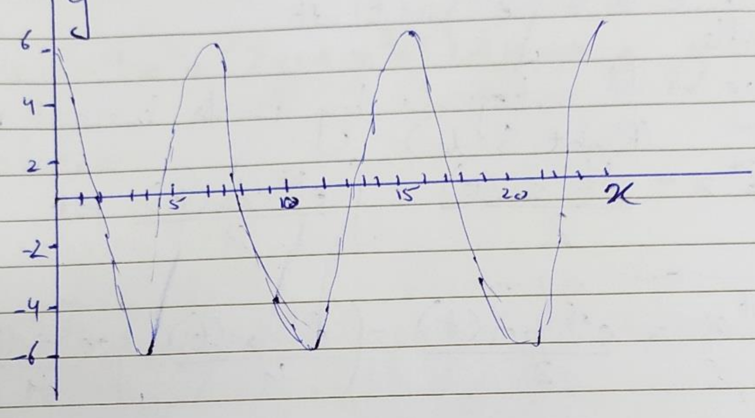
$$C = \frac{3\sqrt{2} + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$C = \frac{3\sqrt{2} + \frac{\sqrt{2}}{2}}{\sqrt{2}} \times 2$$

$$C = 7$$

→ Now putting value of c in equation

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$



(iii) $x' + 2x = \sin t$ —

Solution:

$$x' + 2x = \sin t \longrightarrow A$$

→ As we know that

$$\frac{dy}{dx} + Py = Q$$

→ here $\frac{dy}{dx} = x'$

$$Py = 2x$$

$$Q = \sin t$$

→ where p & q are functions of x or constant

→ For integrating factor we know that

$$I.f = e^{\int p\,dx}$$

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$I.F = e^{2t}$

→ Now multiplying b.s of eq (A) with integrating factors

$$\int \frac{d}{dt} e^{2t} x = \int e^{2t} \sin(t)\, dt$$

→ we will integrate by parts twice in a row (Right Side)

$$\int \frac{d}{dt} e^{2t} x = \frac{e^{2t}\sin(t)}{2} - \int \frac{e^{2t}\cos(t)}{2}\, dt$$

→ Now again

$$\int \frac{d}{dt} e^{2t} x = \frac{e^{2t}\cdot\sin(t)}{2} - \left(\frac{e^{2t}\cos(t)}{4} - \int \frac{-e^{2t}\sin(t)}{4}\, dt\right)$$

→ Now applying linearity.

$$\int \frac{d}{dt} e^{2t} x = \frac{e^{2t}\sin(t)}{2} - \left(\frac{e^{2t}\cos(t)}{4} + \frac{1}{4}\int e^{2t}\sin(t)\, dt\right)$$

→ now integral of $\int e^{2t}\sin(t)\, dt$

$$\int \frac{d}{dt} e^{2t} x = e^{2t}\left(\frac{2\sin(t) - \cos(t)}{5}\right) + C$$

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Q3 Solve the following IVP for the exact equation and find the interval of validity for the solution:

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$, $y(0) = -3$

Solution:

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

→ First we will identify M & N for which we know that

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

→ So $M = 2xy - 9x^2$ & $N = 2y + x^2 + 1$

→ Now for M_y & N_y we know that

$$\frac{dM}{dy} = M \quad \frac{dM}{dy} = 2xy - 9x^2$$
$$= 2x = M_y$$

→ So $M_y = 2x$

→ Now

$$\frac{dN}{dx} = 2y + x^2 + 1 = 2x = N_y$$

So $N_y = 2x$

→ So $M_y = N_y$

→ Now for finding $\Psi(x,y)$ we know that

$$\Psi_x = M \quad \& \quad \Psi_y = N$$

→ Now integrating:

$$\Psi = \int M\,dx \quad \text{or} \quad \Psi = \int N\,dy$$

→ So

$$\Psi(x,y) = \int 2xy - 9x^2\,dx$$

$$\Psi(x,y) = 2y\int x\,dx - 9\int x^2\,dx$$

$$\Psi(x,y) = \cancel{2}y\frac{x^2}{\cancel{2}} - {}^{3}\cancel{9}\frac{x^3}{\cancel{3}} + h(y) \quad \because \text{By power rule}$$

$$\Psi(x,y) = x^2y - 3x^3 + h(y)$$

→ In this case of integration is not constant at all because we are working with two variables

→ To determine $h(y)$ we will use $\Psi_y = N$

Now

→ Differentiate our $\Psi(x,y)$ with respect to y and set is equal to N.

$$\Psi_y = x^2 + h'(y)$$

$$2y + x^2 + 1 = N$$

So

$$2y + 1 = h'(y)$$

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→ Now we can integrate to find h(y)

$$h(y) = \int 2y + 1 \, dy$$

$$h(y) = \frac{\cancel{2}y^2}{\cancel{2}} + y + k$$

$$h(y) = y^2 + y + k$$

→ Now we can write

$$\Psi(x,y) = x^2 y - 3x^2 + y + y + k$$

$$= y^2 + (x^2+1)y + k - 3x^2$$

~~for interval of validity~~

→ As we know $\Psi(x,y) = c$, so

$$y^2 + (x^2+1)y - 3x^3 + k = c$$

$$y^2 + (x^2+1)y - 3x^3 = c - k$$

– Excluding k: $y^2 + (x^2+1)y - 3x^3 = c$ — A

→ Now applying initial condition

$$(-3)^2 + (0+1)(-3) - 3(0)^3 = c$$

$$9 - 3 - 0 = c$$

$$c = 6$$

→ putting value of c in eq A

$$y^2 + (x^2+1)y - 3x^2 = 6$$

$$y^2 + (x^2+1)y - 3x^2 - 6 = 0$$

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→ Solving $y(x)$ using quadratic equation

$$y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^3-6)}}{2(1)}$$

$$= \frac{-(x^2+1) \pm \sqrt{x^2+12x^3+2x^2+25}}{2}$$

→ Now applying initial condition to find

$$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2}$$

$$= \frac{-1 \pm 5}{2} = -3, 2$$

→ So we take "−" and the explicit solution is

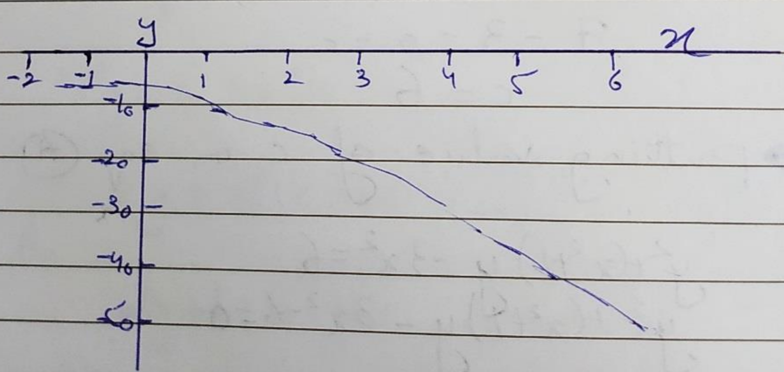
$$y(x) = \frac{-(x^2+1) - \sqrt{x^4+12x^3+2x^2+25}}{2}$$

→ Now for interval of validity

$$x^4+12x^3+2x^2+25 = 0$$

→ By approximating the root using the bisection method

$x = -11.81$ & $x = -1.39$



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ii $\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0, \quad y(5) = 0.$

Solution:

→ In this equation first we will deal with "−" sign. so we will seperate the terms.

$$\frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)\, y' = 0$$

→ Now as we know that $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

→ So here $M = \frac{2ty}{t^2+1} - 2t$ & $N = \ln(t^2+1) - 2$

→ now for M_y & N_t.

$$\frac{dM}{dy} = M = \frac{2ty}{t^2+1} - 2t$$

$$M_y = \frac{2t}{t^2+1}$$

→ Now N_t $\frac{dN}{dt} = N = \ln(t^2+1) - 2$

$$\frac{2t}{t^2+1} = N_t$$

So $M_y = N_t$ which is exact

→ Integrating the first one

$$\Psi(t,y) = \int \frac{2ty}{t^2+1} - 2t\, dt$$

→ Applying linearity

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$$\Psi'(y,t) = 2y\int \frac{t}{t^2+1}\,dt - 2\int t\,dt \rightarrow (K)$$

for $\int \frac{t}{t^2+1}\,dt \rightarrow$ here $u = t^2+1$

$$\frac{du}{dt} = 2t$$

$$dt = \frac{1}{2t}\,du$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

→ Now for $\int \frac{1}{u}\,du$ the standard interval is $\ln(u)$

→ Now plug it on eq (A), we get

$$\Rightarrow \frac{1}{2}\ln(u)$$

$$\Rightarrow \frac{\ln(u)}{2}$$

As $u = t^2+1$, so $\frac{\ln(t^2+1)}{2}$

→ Now for $\int t\,dt$ Applying power rule.

$$\because \int t^n\,dt = \frac{t^{n+1}}{n+1} \text{ with } n=1 = \frac{t^2}{2}$$

→ Now putting the values in eq (K)

$$\cancel{\Psi'(y,t) = \ln(t^2+1) - t}$$

$$\Psi'(y,t) = \frac{\cancel{2}y\ln(t^2+1)}{\cancel{2}} - \frac{\cancel{2}t^2}{\cancel{2}} + h(y)$$

$$\Psi'(y,t) = y\ln(t^2+1) - t^2 + h(y) \quad \text{—— (H)}$$

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→ Now differentiating with respect to y & comparing to N

$$\Psi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2 = N$$

So $h'(y) = -2$

→ Now for $h(y)$ we will re-integrate

$$h'(y) = -2$$
$$\int h'(y) = \int -2$$
$$h(y) = -2y$$

→ So equation (H) becomes

$$\Psi(t,y) = y\ln(t^2+1) - t^2 + (-2y)$$

$$\Psi(t,y) = y\ln(t^2+1) - t^2 - 2y$$

→ So the implicit solution is

$$y\ln(t^2+1) - t^2 - 2y = C \quad \text{— (R)}$$

→ Now applying the given initial condition

$$y(\sqrt{5}) = 0 \text{ and } x = 0$$

So

$$(0)(\ln(\sqrt{5})^2+1) - (\sqrt{5})^2 - 2(0) = C$$
$$-25 = C$$
$$C = -25$$

→ putting value of c in equation (R)

$$y(\ln(t^2+1) - 2) - t^2 = -25$$

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→ Now solving y

$$y(t) = \frac{t^2-25}{\ln(t^2+1)-2}$$

∴ Divided by $\ln(t^2+1)-2)$
∴ subtracted $-t^2$

→ Now for interval of validity

$$\ln(t^2+1)-2=0$$
$$\ln(t^2+1)=2$$
$$(t^2+1)=e^2$$

→ Now subtracting 1 and taking squareroot on b.s

$$\sqrt{t^2+\cancel{1}-\cancel{1}} = \pm\sqrt{e^2-1}$$

$$t = \pm\sqrt{e^2-1}$$

→ so we can have three possible intervals of validity

$$-\infty < t < \sqrt{e^2-1}$$
$$-\sqrt{e^2-1} < t < \sqrt{e^2-1}$$
$$\sqrt{e^2-1} < t < \infty$$

→ The interval of validity is $\sqrt{e^2-1} < t < \infty$ because it contains 5.