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11470
12 semester
Differential Equation

Q1 (a) Differential Equation with 2 examples

It is an equation that Relates one or more functions and their derivatives In applications, the function generally represent physical quantities, the derivatives Represents that Rates of change and the differential equation defines a Relationship b/w the two.

Example: $\frac{d}{dx} + xy = e^{2x}$

$\Rightarrow \quad u_{xx} + u_{yy} + 0$

both contains derivatives so they are differential equations.

(b) Defines Seperable Differential Equation:-

Seperable Differential Equation:-

It is the equation in which equation can be broken into set of seperate equations of lower dimensionality by a method of seperation of variables.

Initial value Problem:-

(i)(a) $y' = \frac{xy^3}{\sqrt{1+x^2}} \quad y(0) = -1$

Sol:- $\int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}} dx$

$= \int y^{-3} dy = \int \frac{x}{\sqrt{1+x^2}} dx$

(2)

$= 1 + x^2 = 0$

$2x\,dx = du$

$\Rightarrow x\,du = \frac{du}{2}$

$\Rightarrow \int y^{-3}\,dy = \int \frac{1}{\sqrt{0}} \frac{du}{2}$

$\Rightarrow \frac{y^{-3+1}}{-3+1} = \frac{1}{2} \frac{0^{-1/2+1}}{-1/2+1} + C$

$\Rightarrow \frac{y^{-2}}{-2} = \frac{2}{2} \sqrt{4} + C$

$= \frac{1}{-2y^2} = \sqrt{1+x} + C$

$\Rightarrow y(0) = -1$

$\Rightarrow \frac{1}{-2(-1)^2} = \sqrt{1} + C$

$\Rightarrow \frac{1}{-2} = 1 + C$

$\Rightarrow -1 \frac{-1}{2} = C$

$= C = \frac{-2-1}{2}$

$C = -3/2$

$y^{-2}/2 = \sqrt{1+x^2} - 3/2$ **Ans.**

(b) $y' = e^{-y}(2x-4) \quad y(5) = 0$ (3)

Sol:-

$$\frac{dy}{dx} = e^{-y}(2x-4)$$

$$\Rightarrow \int \frac{dy}{e^{-y}} = \int (2x-4)\,dx$$

$$\Rightarrow \int e^{y} dy = \frac{2x^2}{2} - 4x + c$$

$$\Rightarrow e^{y} = x^2 - 4x + c$$

$$\Rightarrow y = \ln(x^2 - 4x) + c$$

$$\Rightarrow y = \ln((5)^2 - 4(5)) + c$$

$$\Rightarrow 0 = \ln(25-20) + c$$

$$\Rightarrow 0 = \ln 5 + c$$

$$\Rightarrow c = -\ln 5$$

$$\Rightarrow y = \ln(x^2 - 4x) - \ln 5$$

Q2:- (a) Explain the steps for solving linear Differential Equation

1) Substitute $y = uv$, and

$$dy/dx = u\frac{dv}{dx} + v\frac{du}{dx}$$

into $\frac{dy}{dx} + P(x)y = Q(x)$

2) Factor the parts involving v.

3) Involving v term equal to zero (this gives a D.E in u & x which can be solved in the next step)

4) Solve using separation of variable to find u.

5) Substitute u back into the equation we got at step 2

6) Solve that to find v.

7) Finally substitute u and v into the equation $y = uv$ to get solution.

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ii) $\cos(x)y' + \sin(x)y = 2\cos^2(x)\sin(x) - 1 \quad y[\pi/4] = 3\sqrt{2}$ (4)

$0 \le x \le \pi/2$

Solution:

dividing by $\cos(x)$

$\Rightarrow \quad y' + \dfrac{\sin(x)y}{\cos(x)} = \dfrac{2\cos^2(x)\sin(x) - 1}{\cos(x)}$

$\Rightarrow \quad y' + \tan(x)y = \dfrac{2\cos^3(x)\cdot\sin(x) - 1}{\cos(x)} \quad - (1)$

It has the form $y' + P(x)y = Q(x)$

where $P(x) = \tan x$ & $Q(x) = \dfrac{2\cos^3(x)\cdot\sin(x) - 1}{\cos(x)}$

Integrating Factors

$= e^{\int P(x)\,dx} = e^{\int \tan x\,dx} = \cancel{e}^{\ln(\sec x)}$

$\Rightarrow$ Integrating factor $= \sec x$

Now X'ing (1) by I.f or $\sec(x)$

(1) $\Rightarrow \sec x y' + \sec x \tan(x) y = \dfrac{2\cos^3(x)\sin(x) - 1}{\cos(x)}$

$\Rightarrow \dfrac{d}{dx}[y\sec x] = \dfrac{2\cos^3 x \sin x - 1}{\cos^2 x}$

$\int \cancel{d}[y\sec x] = \int \dfrac{2\cos^3(x)\sin(x) - 1}{\cos^2 x}\,dx$

by solving the integration we get.

$y \sec x = \dfrac{-1}{\tan^2 x + 1} - \tan x + c$

$y = \cos x\left[-\dfrac{1}{\tan^3 x + 1} \cdot \tan x + c\right]$

Now $y(\pi/4) = 3\sqrt{2}$

$\Rightarrow y(\pi/4) = \cos \pi/4\left[\dfrac{-1}{\tan^3(\pi/4) + 1} - \tan(\pi/4) + c\right]$

$\Rightarrow 3\sqrt{2} = \frac{1}{\sqrt{2}}\left[-\frac{1}{2} - 1 + c\right]$ (5)

$3 \times 2 = \frac{-3}{2} + c$

$\boxed{c = -4}$

$y = \cos x\left[-\frac{1}{\tan^2 x + 1} - \tan x - 4\right], \; 0 \le x \le \frac{\pi}{2}$

(iii) $x' + 2x = \sin t$

Solution:-

$x' + 2x = \sin t \quad -(1)$

This D.E has the form $x' + P(t)x = Q(t)$

where $P(t) = 2$ & $Q(t) = \sin(t)$

Integrating factor $= e^{\int P(t)\,dt}$

$I.f = e^{\int 2\,dt} = e^{2t}$

$\times$ing (1) by e^{2t} we get

$= e^{2t}x' + 2xe^{2t} = e^{2t}\sin t$

$= \frac{d}{dt}\left[xe^{2t}\right] = e^{2t}\sin t$

$= \int d\left[xe^{2t}\right] = \int e^{2t}\sin t\,dt$

$= xe^{2t} = \int e^{2t}\sin t\,dt$

Solving by Integrating by Parts method

$\Rightarrow x e^{2t} = \frac{(2\sin t - \cos t)e^{2t}}{5} + c$

$\Rightarrow x = e^{-2t}\left[c + \frac{2e^{2t}\sin t}{5} - \frac{e^{2t}\cos t}{5}\right]$

Q3 (i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = -3$

Sol:- $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$ —(1)

$\Rightarrow (2y + x^2 + 1)\frac{dy}{dx} = -2xy + 9x^2$

$\Rightarrow (2y + x^2 + 1)dy = (9x^2 - 2xy)dx$

$\Rightarrow (9x^2 - 2xy)dx - (2y + x^2 + 1)dy = 0$

$\Rightarrow (9x^2 - 2xy)dx + (-2y - x^2 - 1)dy = 0$

$M(x,y)dx + N(x,y)dy = 0$

$$\frac{\partial M}{\partial y} = -2x$$

$$\frac{\partial N}{\partial x} = -2x$$

So Exact Equation solution exists

$\int M\,dx + \int (\text{term of } N, (\text{free } x))\,dy$

y-axis

$\int (9x^2 - 2xy)dx + \int (-2y - 1)dy = c$

$\Rightarrow \frac{9x^3}{3} - \frac{2x^2y}{2} - \frac{2y^2}{2} - y = c$

$\frac{9x^3}{3} - x^2y - y^2 - y = c$

$y(0) = -3$

$-(-3)^2 - (-3) = c$

$-9 + 3 = c$

$\boxed{c = -6}$

Q3:-
(ii)

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))\, y' = 0 \quad y(5) = 0$$ (7)

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1)) \frac{dy}{dn} = 0$$

$$\left(\frac{2ty}{t^2+1} - 2t\right) dn - (2 - \ln(t^2+1))\, dy = 0$$

$$M(t,y) = \frac{2ty}{t^2+1} - 2t$$

$$N(t,y) = \ln(t^2+1) - 2$$

$$\frac{\partial M}{\partial y} = \frac{2t}{t^2+1}$$

So Exact Equation, solution exists

$$\int M\, dn + \int (\text{term of } N \text{ (free of } n)\, dy$$
y const

$$\int \left(\frac{2ty}{t^2+1} - 2t\right) dt + \int \cdot 2\, dy = c$$

$$y \ln(t^2+1) - t^2 - 2y\, c$$

$$y(5) = 0$$

$$-(5)^2 = c$$

$$\boxed{c = -25}$$